Tatra Mt. Math. Publ. **31** (2005), 75–86

TATRA MOUNTAINS
Mathematical Publications

ON ASYMPTOTIC AND LOGARITHMIC DENSITIES

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ABSTRACT. In the present paper, a construction of a set with prescribed values of lower and upper asymptotic and logarithmic densities is given.

Two important density concepts are the asymptotic and logarithmic density. For their role in the additive or multiplicative number theory the reader is referred to [3]. In general, the density concepts are important tools for measuring of the size of sequences of positive or non-negative integers. One of the most natural questions is whether there are sequences with prescribed admissible values of the chosen density type. An analogous question for a pair of density concepts is posed more seldom. A bit surprising fact is that a result expressing the independence (within admissible bounds of the existence) of the asymptotic and logarithmic densities was proved only recently [4]. In this paper we give a constructive proof of this result (see Theorem 1 below).

For every subset A of $\mathbb{N}$, the set of positive integers, let $\sigma_l(A)$, $\sigma_u(A)$, $\sigma_{ll}(A)$ and $\sigma_{ul}(A)$ be the lower and upper regular asymptotic and logarithmic densities of A, respectively.

In this connection note that a sequence A of positive integers with a positive lower asymptotic density, say d, does not necessarily contain a subsequence $B \subset A$ with asymptotic density d. The set of pairs $\big(\sigma_l(B), \sigma_u(B)\big)$, where B runs over all subsequences of A is described in [1, 2]. To construct examples in which such subsets exist one can also use some results from the theory of distribution functions [7].

Let

$$0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant 1 \tag{1}$$

be real numbers in $[0,1]$ which will remain fixed in what follows.

2000 Mathematics Subject Classification: 11B05.
Keywords: asymptotic density, logarithmic density.

The first author was supported in part by Grants SEP-CONACyT 37259-E, 37260E and UNAM-PAPIIT 104602 and the second author was supported by the Institutional Research Plan AV0Z10300504 and the grant # 201/04/0381 of the Grant Agency of the Czech Republic.

THEOREM 1. *Given $\alpha, \beta, \gamma, \delta \in [0,1]$ with (1), there exists a sequence $A \subset \mathbb{N}$ such that*

$$\sigma_l(A) = \alpha\,, \quad \sigma_u(A) = \delta\,, \quad \sigma_{ll}(A) = \beta\,, \quad \sigma_{ul}(A) = \gamma\,.$$

To start, we point out that the constructed set A is of the form

$$A = \mathbb{N} \cap \left(\bigcup_{n \geqslant 1} [x_n, y_n) \right),$$

for some sequences $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ which range in the set of positive integers or infinity and satisfying $x_n < y_n < x_{n+1}$ for all $n \geqslant 1$. Clearly, $x_n > 1$ for $n \geqslant 2$. In what follows, for a real number $x > 1$ we use $\log x$ for the natural logarithm of x.

For any $n \geqslant 2$ write

$$\sigma_l(n) = \frac{1}{x_n} \sum_{i=1}^{n-1} (y_i - x_i)\,, \tag{2}$$

$$\sigma_u(n) = \frac{1}{y_n} \sum_{i=1}^{n} (y_i - x_i)\,, \tag{3}$$

$$\sigma_{ll}(n) = \frac{1}{\log x_n} \sum_{i=1}^{n-1} \log(y_i/x_i)\,, \tag{4}$$

$$\sigma_{ul}(n) = \frac{1}{\log y_n} \sum_{i=1}^{n} \log(y_i/x_i)\,. \tag{5}$$

The following observation is trivial:

LEMMA 2. *The following equalities hold*

$$\sigma_l(A) = \liminf_n \sigma_l(n)\,, \quad \sigma_u(n) = \limsup_n \sigma_u(n)\,. \tag{6}$$

Assume moreover that

$$\lim_{n\to\infty} \frac{n}{x_n} = 0\,. \tag{7}$$

Then the equalities

$$\sigma_{ll}(A) = \liminf_n \sigma_{ll}(n)\,, \quad \sigma_{ul}(A) = \limsup_n \sigma_{ul}(n) \tag{8}$$

hold.

P r o o f. For a real number $x > 1$ let

$$\sigma_x(A) = \frac{\#\{m \in A\colon m \leqslant x\}}{\#\{m \leqslant x\}} = \frac{\#\{m \in A\colon m \leqslant x\}}{\lfloor x \rfloor}\,.$$

If $y_{n-1} \leqslant x < x_n$ for some $n \geqslant 2$, then

$$\sigma_x(A) = \frac{\sum_{i=1}^{n-1}(y_i - x_i)}{\lfloor x \rfloor} \geqslant \frac{\sum_{i=1}^{n-1}(y_i - x_i)}{x_n - 1} = \sigma_l(n) + O\left(\frac{1}{x_n}\right), \tag{9}$$

while if $x_n \leqslant x < y_n$ then

$$\sigma_x(A) = \frac{\sum_{i=1}^{n-1}(y_i - x_i) + \big(\lfloor x \rfloor - x_n + 1\big)}{\lfloor x \rfloor} \leqslant \frac{\sum_{i=1}^{n}(y_i - x_i)}{y_n - 1} = \sigma_u(n) + O\left(\frac{1}{y_n}\right). \tag{10}$$

Inequalities (9) and (10) clearly show that limits asserted at (6) hold. A similar argument applies to show that $\sigma_{ll}(A) = \liminf_n \sigma'_{ll}(n)$ and $\sigma_{ul}(n) = \limsup_n \sigma'_{ul}(n)$, where

$$\sigma'_{ll}(n) = \frac{1}{\log x_n}\left(\sum_{i=1}^{n-1} \sum_{x_i \leqslant k < y_i} \frac{1}{k}\right) \quad \text{and} \quad \sigma'_{ul}(n) = \frac{1}{\log y_n}\left(\sum_{i=1}^{n} \sum_{x_i \leqslant k < y_i} \frac{1}{k}\right),$$

for all $n \geqslant 2$. The fact that these limits are equal to the ones appearing in (8) follows from the estimate

$$\sum_{s \leqslant k < t} \frac{1}{k} = \log(t/s) + O\left(\frac{1}{s}\right)$$

holding uniformly in positive integers $t > s$, and the fact that (7) is equivalent to the vanishing of the asymptotic density of the sequence $\{x_n\}$, which in turn implies that also the logarithmic density of $\{x_n\}$ is zero. □

Here is how we construct our set A.

Theorem 3. *Let $0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant 1$ be given numbers. Let $(\alpha_n)_{n\geqslant 1}$, $(\beta_n)_{n\geqslant 1}$, $(\gamma_n)_{n\geqslant 1}$, $(\delta_n)_{n\geqslant 1}$ be sequences of real numbers such that the inequalities*

$$\frac{1}{(\log n)^{1/2}} < \alpha_n < \beta_n < \gamma_n < \delta_n < 1 - \frac{1}{(\log n)^{1/2}} \tag{11}$$

hold for all $n \geqslant e^4$, and which converge to α, β, γ, δ, respectively. Put $\lambda_n = 1 + \frac{1}{\log n}$, $u_n = \lambda_n^n$ and $v_n = \lambda'_n \lambda_n^n$, where $\lambda'_n = 1 + \frac{\zeta_n}{\log n}$, with

$$\zeta_n = \begin{cases} \beta_n & \text{if } n \in \big[2^{k^2} + k^4 \log k,\ 2^{(k+1)^2} - (k+1)^4 \log(k+1)\big] \\ & \qquad \text{and } k \equiv 0 \pmod 2,\ k \geqslant 2; \\ \gamma_n & \text{if } n \in \big[2^{k^2} + k^4 \log k,\ 2^{(k+1)^2} - (k+1)^4 \log(k+1)\big] \\ & \qquad \text{and } k \equiv 1 \pmod 2; \\ \delta_n & \text{if } n \in (2^{k^2} - k^4 \log k,\ 2^{k^2} + k^4 \log k) \\ & \qquad \text{and } k \equiv 0 \pmod 2,\ k \geqslant 2; \\ \alpha_n & \text{if } n \in (2^{k^2} - k^4 \log k,\ 2^{k^2} + k^4 \log k) \text{ and } k \equiv 1 \pmod 2. \end{cases}$$

Finally, set $x_n = \lfloor u_n \rfloor$ *and* $y_n = \lfloor v_n \rfloor$. *Then there exists* n_0 *so that the inequalities* $x_n < y_n < x_{n+1}$ *hold for all* $n \geqslant n_0$. *Moreover, if we set*

$$A = \mathbb{N} \cap \left(\bigcup_{n \geqslant n_0} [x_n,\ y_n) \right)$$

then $\sigma_l(A) = \alpha$, $\sigma_{ll}(A) = \beta$, $\sigma_{ul}(A) = \gamma$, *and* $\sigma_u(A) = \delta$.

Proof. We first show that $x_n < y_n < x_{n+1}$ holds for $n > n_0$. Clearly, it suffices to show that the inequalities $v_n - u_n > 1$ and $u_{n+1} - v_n > 1$ hold for $n \geqslant n_0$. There exists $t_0 > 0$ so that the inequality

$$1 + t > e^{t/2} \quad \text{holds for all} \ \ t \in (0, t_0).$$

Thus,

$$v_n - u_n = \frac{\zeta_n}{\log n} \cdot \lambda_n^n \geqslant \frac{\alpha_n}{\log n}\left(1 + \frac{1}{\log n}\right)^n \geqslant \frac{e^{n/(2\log n)}}{(\log n)^{3/2}} \tag{12}$$

holds for $n > e^{1/t_0}$. Clearly, the function of n appearing in the right hand side of (12) tends to infinity with n, therefore the inequality $v_n - u_n > 1$ will certainly hold for n sufficiently large. We now study $u_{n+1} - v_n$. Notice that

$$\begin{aligned}
&\log\left(\frac{u_{n+1}}{v_n}\right) = \log(u_{n+1}) - \log(v_n) \\
&= (n+1)\log\left(1 + \frac{1}{\log(n+1)}\right) - n\log\left(1 + \frac{1}{\log n}\right) - \log\left(1 + \frac{\zeta_n}{\log n}\right) \\
&\geqslant (n+1)\log\left(1 + \frac{1}{\log(n+1)}\right) - n\log\left(1 + \frac{1}{\log n}\right) - \log\left(1 + \frac{\delta_n}{\log n}\right).
\end{aligned} \tag{13}$$

We shall make use of the fact that

$$\log\left(1 + \frac{1}{x}\right) = \frac{1}{x} + O\left(\frac{1}{x^2}\right) \tag{14}$$

holds for large x together with the fact that

$$(x+1)\log\left(1 + \frac{1}{\log(x+1)}\right) - x\log\left(1 + \frac{1}{\log x}\right) = \frac{1}{\log x} + O\left(\frac{1}{(\log x)^2}\right) \tag{15}$$

holds for large x. Indeed, (14) is well-known, while (15) follows from the intermediate value theorem together with the fact that

$$\begin{aligned}
&\frac{d}{dx}\left(x\log\left(1 + \frac{1}{\log x}\right)\right) \\
&\qquad = \log\left(1 + \frac{1}{\log x}\right) - \frac{1}{(\log x)^2} \cdot \frac{1}{(1 + 1/\log x)} = \frac{1}{\log x} + O\left(\frac{1}{(\log x)^2}\right)
\end{aligned}$$

holds for large values of x. Using (14) and (15) in (13), we get

$$\log\left(\frac{u_{n+1}}{v_n}\right) \geqslant \frac{1-\delta_n}{\log n} + O\left(\frac{1}{(\log n)^2}\right) \geqslant \frac{1}{2(\log n)^{3/2}}\,, \tag{16}$$

where (16) holds for large n because of inequality (11.). Thus, for large n we have

$$u_{n+1} \geqslant e^{1/(2\log n)^{3/2}} v_n\,,$$

therefore the inequality

$$u_{n+1} - v_n > v_n\big(e^{1/(2\log n)^{3/2}} - 1\big) > \left(1+\frac{1}{\log n}\right)^n \frac{1}{(2\log n)^{3/2}} > \frac{e^{n/(2\log n)}}{(2\log n)^{3/2}} \tag{17}$$

holds for large enough values of n. The function of n appearing on the right hand side of (17) tends to infinity when n tends to infinity, thus $u_{n+1} - v_n > 1$ certainly holds for n sufficiently large. From now on, we write $n_0 > e^4$ with the meaning understood in the statement of the Theorem 3.

We now look at the densities. Since

$$u_n = \lambda_n^n > e^{n/(2\log n)} > n^2$$

holds for large n, it follows that the condition (7) is fulfilled, so we are entitled to apply Lemma 2. It is also clear that in formulas (2)–(5) we may replace x_n by u_n and y_n by v_n up to an error which is $o(1)$. Indeed, for $n \geqslant n_0 + 1$ we have

$$\begin{aligned}\sigma_l(n) &= \frac{1}{x_n}\sum_{n_0\leqslant i\leqslant n-1}(y_i - x_i) = \frac{1}{x_n}\sum_{n_0\leqslant i\leqslant n-1}(v_i - u_i) + O\left(\frac{n}{x_n}\right)\\ &= \frac{1}{x_n}\sum_{n_0\leqslant i\leqslant n-1}(v_i - u_i) + o(1)\,.\end{aligned}$$

In particular,

$$\frac{1}{x_n}\sum_{n_0\leqslant i\leqslant n-1}(v_i - u_i)$$

is bounded and therefore

$$\begin{aligned}\sigma_l(n) &= \frac{1}{x_n}\sum_{n_0\leqslant i\leqslant n-1}(v_i - u_i) + o(1) = \frac{1}{u_n}\sum_{n_0\leqslant i\leqslant n-1}(v_i - u_i) + O\left(\frac{1}{u_n}\right) + o(1)\\ &= \frac{1}{u_n}\sum_{n_0\leqslant i\leqslant n-1}(v_i - u_i) + o(1)\,.\end{aligned}$$

Similar arguments show that all the estimates

$$\sigma_u(n) = \frac{1}{v_n} \sum_{n_0 \leqslant i \leqslant n} (v_i - u_i) + o(1)\,,$$

$$\sigma_{ll}(n) = \frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \log(v_i/u_i) + o(1)\,,$$

and

$$\sigma_{ul}(n) = \frac{1}{\log v_n} \sum_{n_0 \leqslant i \leqslant n} \log(v_i/u_i) + o(1)$$

hold. From now on, for $n > n_0$ we write

$$\sigma'_l(n) = \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n-1} (v_i - u_i)\,, \quad \sigma'_u(n) = \frac{1}{v_n} \sum_{n_0 \leqslant i \leqslant n} (v_i - u_i)\,,$$

$$\sigma'_{ll}(n) = \frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \log(v_i/u_i)\,, \quad \sigma'_{ul}(n) = \frac{1}{\log v_n} \sum_{n_0 \leqslant i \leqslant n} \log(v_i/u_i)\,,$$

and use the above estimates together with Lemma 2 to conclude that in order to complete the proof of Theorem 3 it suffices to show that $\liminf_n \sigma'_l(n) = \alpha$, $\liminf_n \sigma'_{ll}(n) = \beta$, $\limsup_n \sigma'_{ul}(n) = \gamma$, and $\limsup_n \sigma'_u(n) = \delta$.

We start with the logarithmic means. Notice that

$$\begin{aligned} \frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \log(v_i/u_i) &= \frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \log\left(1 + \frac{\zeta_i}{\log i}\right) \\ &= \frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \frac{\zeta_i}{\log i} + O\left(\frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \frac{1}{(\log i)^2}\right). \end{aligned} \tag{18}$$

Since for a large positive integer n we have

$$\sum_{n_0 \leqslant i \leqslant n} \frac{1}{(\log i)^2} = \int_{n_0}^{n} \frac{dt}{(\log t)^2} + O(1) \ll \frac{n}{(\log n)^2}\,,$$

while $\log u_n \geqslant n/(2\log n)$, we get that the error shown in (18) is $o(1)$. Thus,

$$\frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \log(v_i/u_i) = \frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \frac{\zeta_i}{\log i} + o(1)\,. \tag{19}$$

In particular,

$$\frac{1}{\log u_n} \sum_{n_0 \leqslant i \leqslant n-1} \frac{\zeta_i}{\log i}$$

is bounded. We now use the fact that

$$\log u_n = n \log\left(1 + \frac{1}{\log n}\right) = \frac{n}{\log n} + O\left(\frac{n}{(\log n)^2}\right)$$

to conclude that in (19) we may replace $\log u_n$ by $n/\log n$ creating an error of $o(1)$. All these arguments show that if we take

$$\sigma_{\log}(n) = \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i}{\log i}$$

then

$$\sigma'_{ll}(n) = \sigma_{\log}(n) + o(1)$$

and

$$\sigma'_{ul}(n) = \sigma_{\log}(n) + o(1).$$

Thus, it suffices to compute the lower and upper limits of $\sigma_{\log}(n)$. For this, we claim that for all n we have

$$\frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\beta_i}{\log i} + o(1) \leqslant \sigma_{\log}(n) \leqslant \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\gamma_i}{\log i} + o(1), \tag{20}$$

with equality in both sides occurring infinitely often. To see why this is so, notice that ζ_n takes the extreme values α_n and δ_n in very few n's. Specifically, let n be a very large positive integer. If $i < n$ is such that $\zeta_i \notin \{\beta_i, \gamma_i\}$, then there exists k such that $2^{k^2} - k^4 \log k < n$ and $i \in \left(2^{k^2} - k^4 \log k,\, 2^{k^2} + k^4 \log k\right)$. Clearly, the largest such k is $\ll \sqrt{\log n}$, and so the number of such $i < n$ is certainly not more than

$$\sum_{k \ll \sqrt{\log n}} (2k^4 \log k + 1) \ll \sum_{k \ll \sqrt{\log n}} k^5 \ll (\log n)^3.$$

This argument shows that

$$\frac{1}{n/\log n} \sum_{\substack{n_0 \leqslant i \leqslant n \\ \zeta_i \notin \{\beta_i, \gamma_i\}}} \frac{\zeta_i}{\log i} \ll \frac{(\log n)^3}{n/\log n} = o(1).$$

This shows that the extremal occurrences of $\zeta_i \in \{\alpha_i, \delta_i\}$ do not influence the limiting behavior of $\sigma_{\log}(n)$, and so we get the inequalities asserted at (20) To see that equality in both sides of (20) occurs infinitely often, take, for example,

$n = \lfloor 2^{(k+1)^2} - (k+1)^4 \log(k+1) \rfloor$ with a large even integer k. Then

$$\begin{aligned}\sigma_{\log}(n) &= \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i}{\log i} \\ &= \frac{1}{n/\log n} \sum_{2^{k^2}+k^4 \log k \leqslant i \leqslant n} \frac{\beta_i}{\log i} + \frac{1}{n/\log n} \sum_{n_0 \leqslant i < 2^{k^2}+k^4 \log k} \frac{\zeta_i}{\log i} \\ &= \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\beta_i}{\log i} + O\biggl(\frac{2^{k^2}}{n/\log n}\biggr) \\ &= \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\beta_i}{\log i} + O\biggl(\frac{k^2 \cdot 2^{k^2}}{2^{(k+1)^2}}\biggr) \\ &= \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\beta_i}{\log i} + O\biggl(\frac{k^2}{2^{2k}}\biggr) \\ &= \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\beta_i}{\log i} + o(1)\,,\end{aligned}$$

which proves the equality from the left hand side of (20). The equality on the right hand side of (20) follows similarly by letting n go to infinity through numbers of the form $n = \lfloor 2^{(k+1)^2} - (k+1)^4 \log(k+1) \rfloor$ with an odd integer k. It now suffices to notice that

$$\lim_{n\to\infty} \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\beta_i}{\log i} = \beta \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n/\log n} \sum_{n_0 \leqslant i \leqslant n} \frac{\gamma_i}{\log i} = \gamma\,. \tag{21}$$

Indeed, (21) follows by Cesàro's theorem by noticing, for example, that

$$\lim_{n\to\infty} \frac{\beta_n/\log n}{(n+1)/\log(n+1) - n/\log n} = \lim_{n\to\infty} \beta_n \biggl(\frac{\log(n+1)}{(n+1)\log n - n\log(n+1)}\biggr) = \beta\,,$$

and a similar argument deals with the limit appearing in the right hand side of (21).

Having dealt with the logarithmic means, we now turn our attention to the regular asymptotic means. Notice that

$$\begin{aligned}\sigma_l'(n) = \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n-1} \frac{\zeta_i u_i}{\log i} &= \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i u_i}{\log i} - \frac{\zeta_n}{\log n} \\ &= \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i u_i}{\log i} + o(1)\,.\end{aligned}$$

In particular,

$$\frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i u_i}{\log i}$$

is bounded. Since

$$v_n = u_n + \frac{\zeta_n u_n}{\log n} = u_n + O\left(\frac{u_n}{\log n}\right) = u_n + o(u_n)\,,$$

it follows that in the formula of $\sigma'_u(n)$ we may replace v_n by u_n creating an error of $o(1)$. Hence,

$$\frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i u_i}{\log i} = \sigma'_l(n) + o(1) = \sigma'_u(n) + o(1)\,.$$

Thus, it suffices to investigate the upper and lower limits of the sequence

$$\sigma_{\mathrm{reg}}(n) = \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\zeta_i u_i}{\log i}\,.$$

Since $\alpha_i \leqslant \zeta_i \leqslant \delta_i$ holds for all $i \geqslant n_0$, it is clear that the inequalities

$$\frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\alpha_i u_i}{\log i} \leqslant \sigma_{\mathrm{reg}}(n) \leqslant \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\delta_i u_i}{\log i} \tag{22}$$

hold for all $n > n_0$. It is easy to see, by employing Cesàro's theorem again, that the sequence appearing on the left hand side of inequality (22) tends to α, while the sequence appearing on the right hand side of inequality (22) tends to δ. Indeed, we have

$$\lim_{n\to\infty} \frac{\alpha_n u_n / \log n}{u_n - u_{n-1}} = \lim_{n\to\infty} \alpha_n \left(\frac{1}{\left(1 - \frac{u_{n-1}}{u_n}\right)\log n}\right) = \alpha\,,$$

where here we used the fact that the sequence

$$\begin{aligned}
&\left(1 - \frac{u_{n-1}}{u_n}\right)\log n \\
&\quad = \left(1 - \exp\left((n-1)\log\left(1 + \frac{1}{\log(n-1)}\right) - n\log\left(1 + \frac{1}{\log n}\right)\right)\right)\log n \\
&\quad = \left(1 - \exp\left(-\frac{1}{\log n} + O\left(\frac{1}{(\log n)^2}\right)\right)\right)\log n \\
&\quad = \left(\frac{1}{\log n} + O\left(\frac{1}{(\log n)^2}\right)\right)\log n = 1 + O\left(\frac{1}{\log n}\right)
\end{aligned}$$

tends to 1. It remains to show each one of the estimates

$$\sigma_{\mathrm{reg}}(n) = \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\alpha_i u_i}{\log i} + o(1) \tag{23}$$

and

$$\sigma_{\mathrm{reg}}(n) = \frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant n} \frac{\delta_i u_i}{\log i} + o(1) \tag{24}$$

holds for infinitely many values of n. We shall prove only that (23) holds infinitely often, for the proof of the fact that (24) holds infinitely often is entirely similar. To see (23), let k be a large even integer, write $n = n_k = \lfloor 2^{(k+1)^2} + (k+1)^4 \log(k+1) \rfloor$ and $m = m_k = \lfloor 2^{(k+1)^2} - (k+1)^4 \log(k+1) \rfloor$. Then

$$\sigma_{\mathrm{reg}}(n) = \frac{1}{u_n} \sum_{m \leqslant i \leqslant n} \frac{\alpha_i u_i}{\log i} + O\left(\frac{1}{u_n} \sum_{n_0 \leqslant i \leqslant m} \frac{u_i}{\log i} \right). \tag{25}$$

It suffices to show that the error appearing in the right hand side of (25) is $o(1)$. But this error is certainly

$$\begin{aligned} \ll \frac{u_{m_k}}{u_{n_k}} \sum_{n_0 \leqslant i \leqslant m_k} \frac{1}{\log i} &\ll \frac{u_{m_k}}{u_{n_k}} \cdot \frac{m_k}{\log m_k} \leqslant \frac{u_{m_k}}{u_{n_k}} \cdot \frac{n_k}{\log n_k} \\ &= \left(1 + \frac{1}{\log m_k}\right)^{m_k} \left(1 + \frac{1}{\log n_k}\right)^{-n_k} \frac{n_k}{\log n_k} \\ &= \left(\frac{1 + 1/\log m_k}{1 + 1/\log n_k} \right)^{m_k} \left(1 + \frac{1}{\log n_k}\right)^{m_k - n_k} \frac{n_k}{\log n_k}. \end{aligned} \tag{26}$$

The first factor appearing in the right hand side of (26) is

$$\begin{aligned} \left(\frac{1 + 1/\log m_k}{1 + 1/\log n_k} \right)^{m_k} &= \left(1 + \frac{\log n_k - \log m_k}{\log n_k \log m_k (1 + 1/\log m_k)} \right)^{m_k} \\ &= \left(1 + \frac{\log(n_k/m_k)}{(1+o(1)) \log n_k \log m_k} \right)^{m_k} \\ &< \left(1 + \frac{2(k+1)^4 \log(k+1)}{m_k (1+o(1)) (\log n_k)^2} \right)^{m_k} \\ &\leqslant \exp\left(\frac{2(k+1)^4 \log(k+1)}{(1+o(1)) (\log n_k)^2} \right) \\ &\leqslant \exp\left(\frac{2 \log(k+1)}{(1+o(1)) (\log 2)^2} \right) < k^5, \end{aligned}$$

where for the last inequality we used the fact that $2/(\log 2)^2 < 5$. Thus,

$$\left(\frac{1 + 1/\log m_k}{1 + 1/\log n_k} \right)^{m_k} \frac{1}{\log n_k} \ll k^3. \tag{27}$$

Finally

$$\begin{aligned}\left(1+\frac{1}{\log n_k}\right)^{m_k-n_k} n_k &\leqslant \exp\left(\frac{m_k-n_k}{\log n_k}+\log n_k\right)\\ &= \exp\left(-\frac{2(k+1)^4\log(k+1)}{\big(1+o(1)\big)\cdot k^2\cdot\log 2}+\big(1+o(1)\big)k^2\log 2\right)\\ &= \exp\left(-\big(1+o(1)\big)\frac{2k^2\log k}{\log 2}+\big(1+o(1)\big)k^2\log 2\right) < \exp(-2k^2)\,, \qquad (28)\end{aligned}$$

where the above inequality holds for large values of k because $\log 2 < 1$. From (27) and (28), we get that the right hand side of (26) is bounded above by

$$k^3\cdot\exp(-2k^2)$$

and this last function clearly tends to zero when k tends to infinity. Theorem 3 is therefore proved. □

It is clear that Theorem 3 implies Theorem 1.

In [5] a [6] a relation between asymptotic and logarithmic density is modelled using weighted means in a more general setting for arithmetical semi-groups. In a forthcoming paper we show that under certain assumptions these ideas are also applicable in a modified form in the frame developed in [5].

The authors would like to thank to Oto Strauch for useful comments and careful reading of the original version of the manuscript.

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Received October 20, 2004

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