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FACTORIAL REGULAR REPRESENTATION OF GROUPS IN COMPLETE GRAPHS

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Abstract

A necessary and sufficient condition for the group of isomorphisms involved in a factorization of a complete graph into isomorphic factors is established.

1. Introduction

Factorization of a given complete graph into isomorphic factors attracts the attention for a longer time. An obvious necessary condition is that the number of factors divides the number of edges. Only recently, HARARY, ROBINSON and WORMALD [1], [2] succeeded in proving that this condition is also sufficient. In the present paper factorizations of complete graphs into isomorphic factors are investigated under the hypothesis that the isomorphisms in the question form a group. We introduce the notion of the factorial regular representation of a given abstract group and obtain a necessary and sufficient condition for such representation in the case of complete graphs. Finally we give several corollaries. Although merely special cases of the main result, we hope that they are of independent interest.

2. Definitions and notation

Let G be a graph (finite, without loops and multiple edges) with vertex set V and edges from E. Every permutation f of the vertex set V induces a mapping $f_P : P(V) \to P(V)$, $(P(V) = 2^V)$, and the latter in turn a mapping $f_{P^2} : P(P(V)) \to P(P(V))$. A permutation f of the vertes set V is called an automorphism of the graph G provided $f_P(E) = E$. Given now a partition $\mathfrak{P}$ of the edges of G (into disjoint non-empty sets), an automorphism f of G will be called $\mathfrak{P}$-*admissible* if $f_{P^2}(\mathfrak{P}) = \mathfrak{P}$. Plainly the identity mapping is $\mathfrak{P}$-admissible for every $\mathfrak{P}$ and the set $\Gamma_G(\mathfrak{P})$ of all the $\mathfrak{P}$-admissible automorphisms form a group with respect to the ordinary composition of mappings.

We shall be interested in the following situation:

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If G is a graph and $\mathfrak{P}$ a partition of its edges we say that $\mathfrak{P}$ is a *factorial regular representation* (FRR) of a given abstract group Γ (or that G admits a FRR of Γ) if

(1) Γ is isomorphic to a subgroup of $\Gamma_G(\mathfrak{P})$,

(2) Γ acts on $\mathfrak{P}$ as a regular permutation group, that is, given any two factors F_1 and F_2 of $\mathfrak{P}$, there exists a unique $f \in \Gamma$ with $f_{\mathfrak{P}}(F_1) = F_2$.

Thus, for instance, the well-known factorization of K_{2n} into linear factors is nothing else as a FRR of the cyclic group of order $2n-1$.

Here and in the remainder of the paper always when a permutation property is assigned to an element of Γ it will be understood that this property is shared by the action of this element on G. Further, for undefined group-theoretical notions [4] should be consulted, however with the only exception. Instead of the phrase "Γ is a Frobenius group relative to Γ_1" used in [4] we shall prefer the phrase "Γ_1 is the Frobenius subgroup of Γ". And Γ_1 is the *Frobenius subgroup* of Γ if Γ_1 is a proper subgroup of Γ (i.e., $\{e\} \neq \Gamma_1 \neq \Gamma$) and $\Gamma_1 \cap a\Gamma_1 a^{-1} = \{e\}$ for each $a \in \Gamma - \Gamma_1$.

3. Main result

Our aim is to solve the question when the complete graph K_n admits a FRR of a given abstract group Γ. The corresponding problem for Abelian group was firstly posed and solved by ZELINKA [6]. However Zelinka used different terminology and also our method used here differs entirely from that of Zelinka's paper.

THEOREM. *Let K_n be a complete graph on $n > 1$ vertices and Γ an abstract group. Then K_n admits a FRR of Γ if and only if the order m of Γ is odd and either*

(i) *m divides n or $n-1$, or*

(ii) *there exists the Frobenius subgroup Γ_1 of Γ with*

$$[\Gamma : \Gamma_1] \equiv n \pmod{m}.$$

PROOF. Let a partition $\mathfrak{P}$ of the edges of K_n be a FRR of the group Γ. The factors of $\mathfrak{P}$ are edge-disjoint and therefore Γ acts on the edges of K_n in such a way that the stabilizer of each edge is trivial. This has for us two important consequences:

(j) no element of Γ is involutory, what is equivalent to the fact that the order of Γ is odd, and, secondly,

(jj) the stabilizers Γ_x and Γ_y of any two distint vertices x and y, resp. have only the trivial intersection.

Let $O_1, \ldots, O_{k+1}$ be all the orbits of vertices of K_n with respect to Γ. Consider now the following cases (a), (b), (c):

(a) There exists a vertex x whose stabilizer Γ_x equals Γ. Then one orbit, say O_1, reduces to a singleton and, according to (jj), every vertex other than x necessarily has only the trivial stabilizer. This implies that Γ acts on each of orbits $O_2, \ldots, O_{k+1}$ transitively and regularly. Thus each of these orbits has length equal to the order of Γ, that is m divides $n - 1$.

(b) The case in which every vertex of K_n has the trivial stabilizer gives in a parallel manner that m divides n.

(c) There is a vertex x whose stabilizer Γ_x is a proper subgroup of Γ. Let $x \in O_1$. Since the stabilizers of points on O_1 are conjugtae to Γ_x and conversely, every group conjugate to Γ_x is the stabilizer of a point on O_1, (jj) yields that Γ_x is a Frobenius subgroup of Γ. On the other hand, every group admits uniquely determined Frobenius partition, if any ([4], V.8.17). This implies that only the stabilizers of the vertices on O_1 are proper subgroups of Γ. According to (jj) it is again impossible that in this case Γ is the stabilizer of a vertex of K_n. Therefore the stabilizers of the vertices on the remaining orbits $O_2, \ldots, O_{k+1}$ (if $k \geq 1$) are trivial, which gives that $n = [\Gamma : \Gamma_x] + km$, as required.

To prove the sufficiency suppose firstly that the order m of Γ is odd and that (ii) is satisfied with $n = [\Gamma : \Gamma_1] + km$.

If $k \geq 1$ put $\Gamma_i = \{e\}$ for $i = 2, \ldots, k + 1$. Then divide the vertex set of K_n into $k + 1$ disjoint classes $O_1, O_2, \ldots, O_{k+1}$ in such a way that O_i contains $[\Gamma : \Gamma_i]$ elements $(i = 1, 2, \ldots, k + 1)$. Now label (independently) the elements of O_i with the left cosets of Γ with respect to Γ_i, $i = 1, \ldots, k + 1$.

The embedding of Γ into the group of automorphisms of K_n define in the following manner: mapping $\mathbf{f}$ corresponding to $f \in \Gamma$ translates $g\Gamma_i$ labelled vertex into the vertex with label $fg\Gamma_i$ for every $i = 1, \ldots, k + 1$. Plainly $\mathbf{f}$ is an automorphism of K_n and the mapping $f \mapsto \mathbf{f}$ preserves multiplication. To show that the mapping $f \mapsto \mathbf{f}$ is injective consider the following alternatives:

$k = 0$. If $\mathbf{f} = \mathbf{h}$, then for every $g \notin \Gamma_1$ we have $fg\Gamma_1 = hg\Gamma_1$, i.e., $h^{-1}f \in \in g\,\Gamma_1 g^{-1}$. Since Γ_1 is the Frobenius subgroup of Γ, $h^{-1}f = e$.

If $k \geq 1$ then the previous reasoning works for $\Gamma_2 = \{e\}$.

It follows almost immediately from our construction that the stabilizers of two distinct vertices have only the trivial intersection.

To complete the proof we have to find a partition $\mathfrak{P}$ of edges of K_n on which Γ acts regularly and for which every $\mathbf{f}$ is $\mathfrak{P}$-admissible. Suppose that we have proved that all the edge orbits of K_n with respect ot Γ have the same length, the order of Γ. Then if F is a set of edges, one from each orbit and no two in the same one, the system

$$\mathfrak{P} := \{\mathbf{f}_{p^s}(F);\quad f \in \Gamma\}$$

etablishes the required partition

That all the edge orbits of K_n with respect to Γ have the same length equal to the order of Γ is justified by the following argument: Let $\mathbf{f} \in \Gamma_{(x,y)}$ the stabilizer of the edge (x, y). Then either $\mathbf{f}(x) = x$ and $\mathbf{f}(y) = y$, or $\mathbf{f}(x) = y$ and $\mathbf{f}(y) = x$. In the first case $\mathbf{f} \in \Gamma_x \cap \Gamma_y$, consequently $f = e$. In the second one we have $\mathbf{f}^2 \in \Gamma_x \cap \Gamma_y$ and similarly $f^2 = e$. Since the order of Γ is odd, this is impossible. Thus $\Gamma_{(x,y)}$ is trivial for every edge (x, y) of K_n.

The proof when (i) is satisfied parallels in an obvious manner the preceding lines and therefore can easily be supplied by the reader.

4. Several corollaries

Corollary 1. *If Γ has non-trivial center or if Γ is directly decomposable, then $K_n (n > 1)$ admits a FRR of Γ, if and only if the order of Γ is odd and divides n or $n - 1$.*

Proof. In either case it is possible to give a proof not depending on our Theorem, but we choose the shorter way and show that (ii) cannot occur. Suppose on the contrary that Γ has the Frobenius subgroup Γ_1. Denote

$$\Omega = \Gamma - \bigcup_{g \in \Gamma} g(\Gamma_1 - \{e\})g^{-1},$$

the Frobenius kernel of Γ. Then ([4], V.8.16) if Δ is a normal subgroup of Γ, either $\Delta \subseteq \Omega$ or $\Omega \subset \Delta$.

If Γ has non-trivial center Λ, then $\Lambda \subseteq \Gamma_1$. In opposite case $g\Gamma_1 g^{-1} = \Gamma_1$ for every $g \in \Lambda - \Gamma_1$ which is impossible. On the other hand Λ is a normal subgroup of Γ and so $\Lambda \subseteq \Omega$ or $\Omega \subset \Lambda$. However, both contradict the equality $\Omega \cap \Gamma_1 = \{e\}$([4], V.7.6) and the fact that $\Omega \neq \{e\}$.

Similarly if $\Gamma = A \times B$, then

$$A \subseteq \Omega \quad \text{or} \quad \Omega \subset A,$$

$$B \subseteq \Omega \quad \text{or} \quad \Omega \subset B.$$

Every combination of these possibilities leads again to a contradiction.

This corollary extends the above mentioned result of Zelinka on factorial regular representation of Abelian groups. However in Abelian case our reasoning reduces to a very simple one because a transitive and Abelian action is always regular. The proof of Corollary 1 can be significantly simplified also in the case of nilpotent groups. Namely, if Γ has the Frobenius subgroup Γ_1 then on the one hand the normalizer $N(\Gamma_1)$ of Γ_1 coincides with Γ_1. On the other hand if Γ is finite nilpotent group then ([4], III.2.3.c) every proper subgroup of Γ is properly contained in its normalizer. Thus nilpotent groups cannot posses Frobenius subgroups.

In view of Corollary 1 and the well-known result of FEIT and THOMPSON [1], it is natural to ask whether there are soluble groups of odd order with non-trivial center which are not nilpotent or whether there are soluble groups of odd order without center which are not directly decomposable.

Using results of RÉDEI ([5], Theorems 2—3 and SCHMIDT ([4], III.5.1 and [5], Theorem 1) we can give an affirmative answer even in the class of the minimal non-nilpotent groups, that is groups which every proper subgroup is nilpotent but the groups themselves are not. The necessary results may be stated as follows:

(k) Every minimal non-nilpotent group is soluble and its order has exactly two distinct prime factors.

(kk) The third commutator group Γ''' of a minimal non-nilpotent group Γ is trivial and its second commutator group Γ'' is contained in the center of Γ.

(kkk) If p, q are distinct primes and u a positive integer with $p^u \equiv 1 \pmod{q}$ then there exists a metaabelian group of order $p^u q^v$ for every integer $v \geq 1$. Moreover, if v is even, there exists in addition (exactly one) minimal non-nilpotent group of order $p^u q^v$ which is not metaabelian.

Since a non-abelian group of order pq (where p, q are distinct primes) always possesses the Frobenius subgroup, there are minimal non-nilpotent groups for which the conclusion of Corollary 1 fails. However, under additional hypotheses it can be proved that at least a part of the conclusion of Corollary 1 remains true for every minimal non-nilpotent group. In what follows we shall tacitly assume that the abstract group Γ is isomorphically imbedded in the group of all automorphisms of K_n.

COROLLARY 2. *Let $n > 1$ be not a power of a prime number. Let a minimal non-nilpotent group Γ act transitively on the vertex set of K_n. Then K_n admits a FRR of Γ, if and only if the order of Γ is odd and equals to n.*

PROOF. It is sufficient to thow that Γ acts regularly provided K_n admits a FRR of Γ. Suppose that there is a vertex x of K_n which stabilizer Γ_x is non-trivial. Since Γ acts on K_n transitively, we have $\Gamma_x \neq \Gamma$. Then, as we know from the proof of the Theorem, Γ_x is the Frobenius subgroup of Γ and consequently $N(\Gamma_x) = \Gamma_x$. Moreover, Γ_x is soluble and Γ_x is is nilpotent, in other words Γ_x is the so called Carter's subgroup of Γ. Then ([4], VI.12.3.b) Γ_x is maximal in Γ, which in turn yields that ([4], II.1.4) Γ is primitive. Finally, it follows from Galois' theorem ([4], III.3.2) than then n is a prime power, a contradiction.

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