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NOTES ON DENSITY AND MULTIPLICATIVE STRUCTURE OF SETS OF GENERALIZED INTEGERS

Š. PORUBSKÝ*

The impetus to the development of the theory of logarithmic density was given in a series of papers mainly due to Erdős, Davenport, Behrend and Besicovitch (cf. Chapter V of [4]) in which some properties of the so called primitive sequences were described. Their results can be summarized saying that the logarithmic density is a more sensitive indicator of certain multiplicative properties than the asymptotic one. These results were later extended in various directions and used techniques were adequately refined. In 1967 Alexander

* The work on this paper was done during author's stay as a Fellow of the Alexander von Humboldt Foundation at Fachbereich Mathematik of J.W. Goethe University in Frankfurt/Main (W.Germany).

established that certain generalizations of original results of Erdős and Davenport can be achieved without to intensify the desiderata on the used analytic tools in original Erdős' proofs.

The aim of these notes is twofold. Firstly we describe some of those aspects of relationships amongst various density concepts which reflect - in our opinion - connections between the asymptotic and logarithmic density. To do this we shall use the known model using weighted arithmetic means. Secondly, we will show that Alexander's ideas remain effective also if instead of working with the set of positive integers we recede into a background which depends more on the multiplicative structure of its elements. More concretely, we replace positive integers as the basic set by an *arithmetical semigroup*. This is a free Abelian (multiplicative) semigroup G with the identity element 1 and a countable set P of generators (called the primes of G) on which a real-valued norm mapping $|\ |$ is defined such that

(i) $|ab| = |a||b|$ for all $a,b \in G$,

(ii) the total number $N_G(x)$ of elements $a \in G$ of norm $|a| \le x$ is finite for each real number x.

1. ON LOGARITHMIC DENSITY

Let $m(n)$ be a function defined on an arithmetical semigroup G and taking positive real values. Then given a subset C of G we put

$$\sigma_x(C,m) = \frac{\sum\limits_{|n|\le x} m(n)\chi(n)}{\sum\limits_{|n|\le x} m(n)}$$

to denote the m-weighted arithmetic means of the indicator χ of C. The numbers

$$\underline{\sigma}(C,m) = \liminf_{x\to\infty} \sigma_x(C,m)$$

$$\overline{\sigma}(C,m) = \limsup_{x\to\infty} \sigma_x(C,m)$$

will be called, as expected, the *lower m-density* of C, and the *upper m-density* of C, resp. In the case $\underline{\sigma}(C,m)= = \overline{\sigma}(C,m)$ this common value is called the *m-density* $\sigma(C,m)$ of C.

In what follows we shall need the following result on the consistency of two weighted arithmetic means.

LEMMA 1. *Let* m, s *be two positive functions defined on* G *such that*

(*i*) *both series* $\sum_{n\in G} m(n)$ *and* $\sum_{n\in G} s(n)$ *are divergent,*

(*ii*) *for all* n_1, n_2 *in* G *the inequality* $|n_1| \leq |n_2|$ *implies*

$$\frac{m(n_2)}{m(n_1)} \geq \frac{s(n_2)}{s(n_1)} .$$

Then for every subset C *of* G *we have*

$$\underline{\sigma}(C,m) \leq \underline{\sigma}(C,s) \leq \bar{\sigma}(C,s) \leq \bar{\sigma}(C,m).$$

If G is the set of all positive integers this is a konwn variant of the standard Knopp's kernel theorem and a proof can be found e.g. in [6]. However, also our more general formulation can be easily derived from this classical one.

In order to have a possibility to work reasonably with this m-density we have to postulate some of its properties which seem to be important for our purposes. These are as follows:

I. *the series* $\sum_{n\in G} m(n)$ *diverges,*

II. *to every* $a \in G$ *there exists a positive real number* $\hat{m}(a)$, $\hat{m}(a)<1$ *if* $a \neq 1$, *such that for every subset* C *of* G *having* m*-density* $\sigma(C,m)$ *the set*

$$aC = \{ac : c \in C\}$$

has also m*-density* $\sigma(aC,m)$ *and*

$$\sigma(aC,m) = \hat{m}(a)\sigma(C,m),$$

III. *the series* $\sum_{n \in G} \hat{m}(n)$ *diverges.*

Condition I is obviously equivalent to the requirement that in the associated m-density points are to have zero density. This condition guarantees simultaneously that our m-weighted arithmetic means form a regular transformation and a classical result of STEINHAUS [7] asserts that there is always a sequence of zeros and ones not summable by our m-weighted meands, in other words, there exists a sequence not having m-density.

Condition II should reflect certain properties of the multiplicative structure of G. It is trivially satisfied by the asymptotic (and consquently also by the logarithmic) density. The function $\hat{m}$ is plainly completely multiplicative.

As to condition III, this is not completely independent of both previous conditions I and II as the following lemma shows.

LEMMA 2. *If*

$$\frac{\sigma_x(pG,m)}{\hat{m}(p)} - 1 < c,$$

c a constant, for all x and $p \in P$, then conditions I *and* II *imply* III.

PROOF. Firstly observe that along the standard argument it can be proved that the m-density of the set $G_{\langle p_1 \ldots p_k \rangle}$ of all the elements of G which are coprime to the product $p_1 \ldots p_k$ of primes $p_1, \ldots, p_k$ is equal to

$$\sigma(G_{\langle p_1 \ldots p_k \rangle}, m) = \prod_{i=1}^{k} (1-\hat{m}(p_i)).$$

Now suppose that $\sum_{n \in G} \hat{m}(n) < \infty$. Then the same holds for the series $\sum_{p \in P} \hat{m}(p)$. Proceding as in WEGMANN's proof of Satz 2.2 [8] we obtain that the m-density of the set of all the elements coprime to every prime of P is equal to the product

$$\prod_{p \in P} (1-\hat{m}(p)).$$

On the other hand, the m-density of this set containing only the identity element 1 is obviously zero. This contradiction leads to the conclusion of Lemma 2.

The remark after the above mentiond Satz 2.2 in WEGMANN's paper [8] shows that in general condition III is not a consequence of I and II even in the case of asymptotic density (i.e. $m(n)=1$ for all $n \in G$). However, Wegmann's example is far beyond of our considerations in the next part of our paper.

As mentioned earlier, function $\hat{m}(n)$ is completely multiplicative. Since $0<\hat{m}(n)<1$ for $n \neq 1$,

$$\lim_{|n| \to \infty} \hat{m}(n) = 0.$$

Thus the function

$$n \to \frac{1}{\hat{m}(n)}$$

defines a norm mapping on G. Then condition III is a statement on the zeta function

$$\hat{\zeta}_G(s) = \sum_{n\in G} \left(\frac{1}{\hat{m}(n)}\right)^{-s} = \sum_{n\in G} \hat{m}(n)^s$$

of G with respect to this new norm mapping. Namely that $s=1$ is not a point of analyticity of $\hat{\zeta}_G$. From this point of view it is not surprising that the conditon III is not always satisfied. However in some cases there are connections between both zeta's

$$\zeta_G(s) = \sum_{n\in G} |n|^{-s} \quad \text{and} \quad \hat{\zeta}_G(s) = \sum_{n\in G} \hat{m}(n)^s.$$

So for instance, if m is completely multiplicative and

$$\sum_{|n|\leq x} m(n) = x^{\Delta} L(x),$$

where $L(x)$ is a slowly oscilating function, then

$$\hat{m}(n) = m(n)|n|^{-\Delta}.$$

Thus if $m(n)=1$ for all $n\in G$ we have even the equality

$$\hat{\zeta}_G(s) = \zeta_G(s\Delta)$$

provided G is Δ-regular in the sense of WEGMANN [8] (i.e. $N_G(x)=x^{\Delta}L(x)$, L slowly oscilating). And the above mentioned counter-example due to Wegmann gives a 1-regular semigroup for which $\sum_{n\in G} |n|^{-1} < \infty$.

PROPOSITION 1. *Let $m(n)$ be a completely multiplicative function on an arithmetical semigroup G such that*

$$\sum_{|n|\le x} m(n) = Bx^{\Delta} + O(x^{\Xi}) \quad \text{as} \quad x\to\infty.$$

Then

1. $$\sum_{|n|\le x} \hat{m}(n) = \Delta B \log x + \Psi_m + O(x^{\Xi-\Delta})$$

   with a suitable Ψ_m,

2. $\underline{\sigma}(C,m) \le \underline{\sigma}(C,\hat{m}) \le \overline{\sigma}(C,\hat{m}) \le \overline{\sigma}(C,m)$

   for every $C\subset G$.

PROOF. Since $\sum_{|n|\le x} m(n) = x^{\Delta}L(x)$, where $L(x)$ is slowly oscilating, the previous remarks yield that $\hat{m}(n) = m(n)|n|^{-\Delta}$, and 1. follows by partial summation. This simultaneously shows that condition III is fulfilled. Then Lemma 1 completes the proof.

From the point of view of analytic number theory the most interesting arithmetical semigroups are those satisfying the so called Axiom A (after KNOPFMACHER [5]):

AXIOM A: *There exist positive constants* A *and* δ, *and a constant* η *with* $0\leq\eta<\delta$, *such that*

$$N_G(x) = Ax^{\delta} + O(x^{\eta}) \quad as \quad x\to\infty.$$

If semigroup G satisfies Axiom A we can define the lower and upper logarithmic density of a set C as follows:

$$\underline{l}(C) = \liminf_{x\to\infty} \frac{1}{\delta A \log x} \sum_{\substack{a\in C \\ |a|\leq x}} |a|^{-\delta},$$

$$\bar{l}(C) = \limsup_{x\to\infty} \frac{1}{\delta A \log x} \sum_{\substack{a\in C \\ |a|\leq x}} |a|^{-\delta}.$$

It is clear from our previous notes that this definition coincides with the lower and upper $\hat{m}$-densitites provided the original m-density is the asymptotic one and G satisfies Axiom A. If we further denote $\underline{d}(C) = \underline{\sigma}(C,1)$ and $\bar{d}(C) = \bar{\sigma}(C,1)$ the lower and upper asymptotic densities of a set C, then we have proved also the following important inequalities:

COROLLARY. *If an arithmetical semigroup* G *satisfies Axiom* A *then*

$$0 \leq \underline{d}(C) \leq \underline{l}(C) \leq \bar{l}(C) \leq \bar{d}(C) \leq 1$$

for every subset C *of* G.

Thus if G satisfies Axiom A then the asymptotic density leads to a "thiner" one, namely to the logairthmic density. However reproducing this procedure once more, now starting with the logarithmic density instead of with the asymptotic one, we do not obtain again a "thiner" new density in general. Namely, if G satisfies Axiom A the resulting density is again the logarithmic one. This follows from the fact that if m is a completely multiplicative function on G such that

$$\sum_{|n| \leq x} m(n) = L(x),$$

where L is slowly oscilating, then

$$\hat{m}(n) = m(n)$$

for all $n \in G$. This gives the following simple result:

PROPOSITION 2. *Let* $m(n)$ *be a completely multiplicative function on an arbitrary arithmetical semigroup* G *such that*

$$\sum_{|n| \leq x} m(n) = Bx^{\Delta} + O(x^{\Xi}) \quad \text{as} \quad x \to \infty.$$

Then

$$\hat{\hat{m}} = \hat{m}.$$

The next result shows that every m-density which include Cesaro's first means leads to the logarithmic density.

PROPOSITION 3. *If* G *satisfies Axiom* A *and the function* m *satisfies the following two conditions*

1. $\sum_{n \in G} m(n)$ *diverges,*

2. *if* $|n_2| \leq |n_2|$ *then* $m(n_1) \leq m(n_2)$*,*

then the lower and upper $\hat{m}$*-density coincides with the* *lower and upper logarithmic density, resp.*

PROOF. From Lemma 1 there follows that if a set has m-density it has also the asymptotic one and the result follows.

2. ON ALEXANDER'S MULTIPLICATIVE DECOMPOSITION

In this section we shall closely follow ALEXANDER's ideas from [1] and therefore proofs will be omitted. We refer the reader to [1] for details.

The failure of the assertion in condition III in general goes also back to the fact that the m-density is not a countably additive measure and thus neither continuous from above at $\emptyset$ nor continuous from below at every set with m-density. In this direction the following lemma can be proved similarly as in [1].

LEMMA 3. *Suppose that* $\{E_i\}_{i=1}^{\infty}$ *is a sequence of sets each having m-density such that also every* $F_j = \bigcup_{i \le j} E_i$ *has m-density. If there exists a convergent series* $\sum_{i=1}^{\infty} L_i$ *with positive terms and*

$$\sigma_x(E_i, m) \le L_i$$

for all i *and* x, *then*

$$\sigma\left(\bigcup_{i=1}^{\infty} E_i, m\right) = \lim_{j\to\infty} \sigma(F_j, m).$$

In what follows unless contrary is stated we shall always suppose that arithmetical semigroups under considerations satisfy the above mentioned Axiom A.

One of the most known non-arithmetical examples of an arithmetical semigroup satisfying Axiom A is the category A of all the finite Abelian groups. Here the multiplication is the usual direct product, norm function $|A| = \mathrm{card}(A)$, the primes are various cyclic groups of prime power order. As shown by ERDŐS and SZEKERES [3]

$$N_A(x) = \left[\prod_{k=2}^{\infty} \zeta(k)\right]x + O(x^{1/2}).$$

A lot of results known in the classical analytic number theory carry over to arithmetical semigroups satisfying Axiom A. So for instance zeta function $\zeta_G(s)$ of such semigroups has a simple pole at $s=\delta$, or we have the following result as a consequence of the generalized Mertens' formula ([5], Lemma 6.3.1):

LEMMA 4. *With suitable constants* M_1 *and* M_2 *one has*

$$M_1 \log x < \prod_{|p| \in x} (1-|p|^{-\delta})^{-1} < M_2 \log x.$$

Here the constants M_1, M_2 are absolute in the sense that they depend on A, δ and η but not on the particular semigroup satisfying Axiom A.

If we denote after Alexander, $P(x)$ the set of all the elements of G that are composed entirely of primes in norm greater than x, we can prove:

LEMMA 5. *If* $a \in G$, *then*

$$d(aP(x)) = |a|^{-\delta} \prod_{|p| \le x} (1-|p|^{-\delta}).$$

Further, let Γ be the family of all arithmetic functions f with

$$f(n) \ge g(n) := \max\{|p| : p \in P,\ p \text{ divides } n\}$$

for every $n \in G$. If C is a subset of G and $f \in \Gamma$, then the f-primary part $A(f,C)$ of C is defined as

$$A(f,C) = \{c \in C : c \notin bP(f(b)) \text{ for all } b \neq c,\ b \in C\}$$

The secondary part $B(f,C)$ of C is

$$B(f,C) = C - A(f,C).$$

THEOREM 1. *Let* $c \in C$ *and* $f \in \Gamma$. *Then either* c *belongs to* $A(f,C)$ *or* c *may be uniquely represented as* $c = as$, *where* a *belongs to* $A(f,C)$ *and* s *belongs to* $P(f(a))$.

In [2] ERDŐS proved the "classical" result that if C is the so called primitive sequence of positive integers, then the series

$$\sum_{a \in C} \frac{1}{a \log a}$$

converges. The next theorem contains this result and its later generalization due to ALEXANDER [1].

THEOREM 2. *Let* C *be a subset of* G *and* $f \in \Gamma$. *Then*

$$\sum_{a \in A(f,C)} \frac{1}{|a|^{-\delta} \log f(a)} \leq M_2.$$

This result has an interesting corollary closely related to the original Erdős result. Let G_m be the multiplicative semigroup of associated integers of a complex Euclidean quadratic field $Q(\sqrt{m})$. Then G_m satisfies Axiom A with $\delta=1$ while the norm is the square of the usual absolute value. Since $|\log a| \geq \log|a|$ for $a \in G_m$ (here $|\ |$ denotes the absolute value of a complex number), we have:

COROLLARY. *Let* C *be a sequence of integers in a complex Euclidean quadratic field* $Q(\sqrt{m})$, *i.e.* $m \in \{-1, -2, -3, -7, -11\}$. *If* $f(n)=|n|^2$ *then the series*

$$\sum_{a \in A(f,C)} \frac{1}{a^2 \log a}$$

converges.

If Γ' is the set of those $f \in \Gamma$ for which there exists a real number $K=K(f)$ such that

$$f(n) = |n|^K$$

for all $n \in G$, then we have:

THEOREM 3. *Let* C *be an arbitrary subset of* G.

If f belongs to Γ', *then*

$$l(A(f,C)) = 0.$$

Let Ω be the family of all arithmetical functions h defined on G for which $h(n)\log|n|$ is positive and non-decreasing and for which

$$\sum_{n\in G} (h(n)|n|^{\delta}\log|n|)^{-1}$$

is a divergent series. Then Lemma 1 gives

$$\underline{l}(C) \leq \underline{\sigma}(C,(h(n)|n|^{\delta}\log|n|)^{-1}) \leq$$

$$\leq \bar{\sigma}(C,(h(n)|n|^{\delta}\log|n|)^{-1}) \leq \bar{l}(C)$$

for every subset C of G.

Let Γ'' be the set of those $f\in\Gamma$ for which

$$f(n) \leq |n|^{h(n)}$$

for some $h\in\Omega$. Then we have:

THEOREM 4. *Let* $f\in\Gamma''$ *and* C *a subset of* G . *Then*

$$\underline{l}(A(f,C)) = \sigma(A(f,C),\ (h(n)|n|^{\delta}\log|n|)^{-1}) = 0.$$

The next result is a rewritten form of Alexander's principal result on the density of $B(f,C)$.

THEOREM 5. *Let* C *be a subset of* G *and* $f \in \Gamma$. *Suppose that* $l\{C \cap aP(f(a))\}$ *exists for each* $a \in A(f,C)$. *Then* $l(B(f,C))$ *exists and*

$$l(B(f,C)) = \sum_{a \in A(f,C)} l\{C \cap aP(f(a))\}.$$

Thus seeing that all fundamental theorems of ALEXANDER's paper [1] on decomposition carry over to arithmetical semigroups satisfying Axiom A, it is possible to rewrite also all the remaining theorems of [1] which are based on the above mentioned results. So for instance, if C has the property that each member of C divides only finitely many members of C, then $l(C)=0$. Or, if $K_1, K_2, \ldots$ is any sequence of positive numbers and if C does not posses zero logarithmic density, then C contains a division chain of the form q_1, $q_1 q_2$, $q_1 q_2 q_3$, $,\ldots,$ where q_{i+1} is composed entirely of primes greater than $|q_1 \cdots q_i|^{K_i}$, etc., etc. We refer the reader to [1] for the remaining results.

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Š. Porubský
Mathematical Institute
Slovak Academy of Sciences
ul. Obrancov mieru 49
814 73 Bratislava, Czechoslovakia