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FUNCTIONS THAT PRESERVE UNIFOM DISTRIBUTION (PRELIMINARY ANNOUNCEMENT)

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This is a preliminary report on a joint work with T. Šalát and O. Strauch and a more detailed version will be published elsewhere.

We shall use the following basic notation:
I stands for the interval $\langle 0, 1\rangle$,
M_I denotes the system of the functions from I to I,
R_I is the system of the Riemann-integrable functions from M_I.

We shall study properties of functions $f \in M_I$ sharing the property that for every uniformly distributed sequence $\{x_n\}_{n=1}^{\infty}$ of numbers from I also that sequence $\{f(x_n)\}_{n=1}^{\infty}$ is uniformly distributed in I. We shall denote by T_I the system of all such functions.

The starting result is the following one:

Theorem 1. A function $f \in T_I$ belongs to T_I if and only if for every function $g \in R_I$ also the composition $g \circ f$ belongs to R_I and

$$\int_0^1 g(x)\,\mathrm{d}x = \int_0^1 g(f(x))\,\mathrm{d}x\,.$$

The same proof technique leads to the expected modification:

Theorem 1'. A function $f \in M_I$ belongs to T_I if and only if for every $g \in C_I$ also $g \circ f \in R_I$ and

$$\int_0^1 g(x)\,\mathrm{d}x = \int_0^1 g(f(x))\,\mathrm{d}x\,.$$

The function $g(x) = x$ gives that every function in T_I is Riemann-integrable. However for such functions the following surprising result can be proved.

Theorem 2. Let $f \in R_I$. Then f belongs to T_I if there exists a uniformly distributed sequence $\{x_n\}_{n=1}^{\infty}$ in I such that $\{f(x_n)\}_{n=1}^{\infty}$ is also uniformly distributed.

The function

$$f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ otherwise} \end{cases}$$

shows that there are functions $f \in R_I$ which preserve infinitely many uniformly distributed sequences and simultaneously for infinitely many not uniformly distributed sequences $\{x_n\}_{n=1}^{\infty}$ the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is not uniformly distributed.

Some additional conditions on functions f from T_I reduce considerably the possibilities for f. Here are some of them:

Theorem 3. Let $f \in T_I$. If f is continuous and injective then either $f(x) = x$ or $f(x) = 1 - x$.

Theorem 4. If a function $f \in T_I$ has the derivative at every point interior to I then either $f(x) = x$ or $f(x) = 1 - x$.

Theorem 5. A Darboux function $f \in M_I$ belongs to T_I if and only if for every $x, x' \in I$ we have

$$|f(x) - f(x')| = |x' - x|.$$

A sufficient condition is given in the next result.

Theorem 6. If a function $f \in M_I$ has the property that

$$\max_{n \to \infty} (f(x_n) - x_n) = 0$$

for every sequence $\{x_n\}_{n=1}^{\infty}$ of numbers from I then $f \in T_I$.

There is a close connection between the functions $f \in R_I$ and the functions measurable in the Jordan sense. In this direction the following result can be proved.

Theorem 7. A function $f \in M_I$ belongs to T_I if and only if f is measurable in the Jordan sense and if

$$|f^{-1}(E)| = |E|$$

for every interval $E \subset I$.

This theorem enables us to construct examples of functions from T_I. The most simple of them are the zigzag functions with the height of every tooth exactly equal to one.

The closet generalization of zigzag functions are the piecewise linear functions. Let f be such a function. Let

$$f^{-1}(y) = \{x_1, x_2, \ldots, x_k\}$$

be the so called level set at $y \in i$. The function f has the derivative at every point of the level set for all but a finite set of points $y \in I$. It can be proved for instance:

Theorem 8. A piecewise linear function f belongs to T_I if and only if

$$\sum_{x_i \in f^{-1}(y)} \frac{1}{|f'(x_i)|} = 1$$

for every such $y \in I$ for which all the required derivatives exist.

This result in a rewritten form can be used for a construction of the all piecewise linear functions from T_I.

If we endow the set M_I with the supremum metric then it can be easily seen that the set T_I is closed in M_I. The topological properties of T_I in M_I culminates in the next theorem.

Theorem 9. The set T_I is perfect and nowhere dense in M_I.

The last result we mention is of a more specific nature. We first recall the notion of the somewhat continuous function. A function f from the topological space X to a topological space Y is somewhat continous if for every open set V in Y the condition $f^{-1}(V) \neq \emptyset$ implies that the interior $\operatorname{Int} f^{-1}(V)$ is also nonempty. Then Theorem 7 implies in turn that:

Theorem 10. Every function f from T_I is somewhat continuous.

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SÚHRN

FUNKCIE ZACHOVÁVAJÚCE ROVNOMERNÉ ROZDELENIE

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V práci sa podávajú vybrané výsledky o funkciách f: $\langle 0, 1\rangle \mapsto \langle 0, 1\rangle$, ktoré prevádzajú každú rovnomerne rozdelenú postupnosť $\{x_n\}_{n=1}^{\infty}$ do rovnomerne rozdelenej postupnosti $\{f(x_n)\}_{n=1}^{\infty}$.

РЕЗЮМЕ

ФУНКЦИИ СОХРАНЯЮЩИЕ РАВНОМЕРНОЕ РАСПРЕДЕЛЕНИЕ

Штефан Порубски, Братислава

Описываются избранные результаты касающеися функций f из единичного интервала $\langle 0, 1\rangle$ в единичный интервал, которые переводят каждую равномерно распределенную последовательность $\{x_n\}_{n=1}^{\infty}$ в $\langle 0, 1\rangle$ в равномерно распределенную последовательность $\{f(x_n)\}_{n=1}^{\infty}$.