IDENTITIES WITH COVERING SYSTEMS AND APPELL POLYNOMIALS

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Appeared in: *Proceedings "Number Theory in Progress" of the International conference on number theory in honor of the 60th birthday of Andrzej Schinzel, Walter de Gruyter, Berlin – New York 1999, pp. 407–417*

With the deepest admiration to Professor Andrzej Schinzel on the occasion of his sixties

Abstract. We show how based on generalized multiplication theorems for Appell sets of polynomials various identities involving some values of these polynomials and offsets and moduli of finite systems of arithmetical progressions can be derived in a uniform manner, thereby generalizing some recent and older recurrences for Bernoulli and Euler polynomials.

1. Introduction and preliminaries

Let $a \pmod b$ represent the arithmetic progression $\{n; n = a + kb, k \in \mathbb{Z}\}$ and the Iverson brackets $[x \in a \pmod b]$ its indicator. Given a weight function $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ with complex values, a finite collection (i.e. the repetitions of progressions are allowed) of arithmetic progressions

$$\{a_i \pmod{b_i}; 1 \le i \le k\}, \tag{1}$$

will be called a μ–**system** if μ_i denotes the weight assigned to the ith class $a_i \pmod{b_i}$, $i = 1, 2, \ldots, k$. The a_i's will be called **offsets** and the b_i's the **moduli** of the system (1). We shall always suppose that the offsets are **standardized**, i.e. $0 \le a_i < b_i$. The function

$$\mathfrak{m}(n) = \sum_{i=1}^{k} \mu(i)[n \in a_i \pmod{b_i}],$$

will be called the **covering function** of the μ–system (1). To simplify the notation we shall also use the abbreviation that (1) is a $(\mu, \mathfrak{m})$–**cover**. This notation introduced in [Poru1975] covers several previously investigated notions:

- If $\mu_i = 1$ for $i = 1, \ldots, k$ and $\mathfrak{m}(n) \ge 1$ for every $n \in \mathbb{Z}$, i.e. if each integer belongs to at least one of the arithmetic progressions (1), then (1) is called a **cover**. This notion was introduced by P.Erdős at the beginning of thirties under the additional restriction that the moduli are distinct. However, we shall not adopt this convention about the moduli.
- If $\mu_i = 1$ for $i = 1, \ldots, k$ and $\mathfrak{m}(n) = m, m \in \mathbb{N}$ for every $n \in \mathbb{Z}$, the resulting cover is called an **exact** m–**cover**. This notion was introduced in [Poru1976]. The most important special cases of exact m–covers are exact 1–covers, or simply **exact covers**, a notion also going back to Erdős[1].

1991 *Mathematics Subject Classification.* 11B25, 11B68, 11B83, 33E30.

Research supported by Grant # 201/97/0433 of the Grant Agency of the Czech Republic. The author would like to express his sincere thanks the organizers of conference for the hospitality.

[1] In the past names like **disjoint cover** or **exactly covering system** were also used.

Exact covers or more generally the exact m–covers with $m \geq 1$ are the most closest generalizations of the systems of arithmetic progressions representing the complete residue systems

$$0 \pmod{k}, 1 \pmod{k}, \dots, k-1 \pmod{k}. \tag{2}$$

Note that there are 2–covers which [Poru1976, Choi's example] cannot be written as a collection of two exact covers. More generally [Zhan1991], for every $m = 2, 3 \dots$ there exists and exact m–cover no subset of which is an exact n–cover with $0 < n < m$. (we refer the reader to [Poru1981] for more details on results about systems of arithmetical progressions up to 1981).

Covering function $\mathfrak{m}$ of a μ–system is a periodic function and its (smallest positive) period will be denoted by b_0 in what follows. The period b_0 is obviously a divisor of $M = \text{l.c.m.}\{b_i; 1 \leq i \leq k\}$.

In 1973 A.S.Fraenkel [Frae1973] proved (the case $x = 0$ and $(\mu, \mathfrak{m})$–cover being an exact cover in (3) below) a characterization of exact covers in terms of values of Bernoulli polynomials at arguments depending on the offsets and moduli of system (1). Fraenkel's results gave impetus to a series of various generalizations. E.g. in [Poru1975, Theorem 2] and [Por1994a, Theorem 1] it was proved that: *A system (1) is a $(\mu, \mathfrak{m})$–cover if and only if*

$$\sum_{t=1}^{k} \mu_t b_t^{r-1} B_r\left(\frac{x+a_t}{b_t}\right) = b_0^{r-1} \sum_{t=0}^{b_0-1} \mathfrak{m}(t) B_r\left(\frac{x+t}{b_0}\right) \tag{3}$$

for every $r \in \{0, 1, 2, \dots\}$ and fixed real x, where $B_r(x)$ stands for the rth Bernoulli polynomial [2], $r = 0, 1, 2, \dots$.

The "only if" part of this result can be proved in an elementary way using the distribution property of Bernoulli polynomials known as the *Raabe's multiplication formula* for Bernoulli polynomials

$$B_r(x) = m^{r-1} \sum_{a=0}^{m-1} B_r\left(\frac{x+a}{m}\right) \tag{4}$$

holding for any real number x and integers $r \geq 0, m \geq 1$. This multiplication formula plays the prototype role of those investigated in this paper.

Deeba and Rodriguez [DeRo1991] and independently some other authors (c.f. [Gess1989]), proving a conjecture of V.Namias [Nami1986], proved the following recurrence for the Bernoulli numbers $B_r = B_r(0)$ and positive integers $n > 1$

$$B_r = \frac{1}{n(1-n^r)} \sum_{s=0}^{r-1} n^s \binom{r}{s} B_s \sum_{t=1}^{n-1} t^{r-s}, \quad r = 1, 2, \dots \tag{5}$$

J.Beebee [Beeb1992] extended this recurrence to the form where the offsets and the modulus of the system (2) are replaced by offsets and moduli of an arbitrary exact cover. More precisely, (for a generalization to $(\mu, \mathfrak{m})$–covers see [Por1994a]): *System (1) is an exact cover if and only if* $\sum_{t=1}^{k} b_t^{-1} = 1$ *and*

$$B_r = \frac{1}{1 - \sum_{j=1}^{k} b_j^{r-1}} \sum_{s=0}^{r-1} \binom{r}{s} B_s \sum_{t=1}^{k} b_t^{r-1} \left(\frac{a_t}{b_t}\right)^{r-s} \tag{6}$$

for every positive integer r.

Motivated by a long standing Erdős – Selfridge conjecture for covers with distinct moduli (c.f. [Poru1981] for more details), seemingly unrelated identities which take the parity of the moduli of a $(\mu, \mathfrak{m})$–cover (1) into account but involve instead of

[2] E.g. defined by the generating series $ze^{xz}/(e^z - 1) = \sum_{r=0}^{\infty} B_r(x) z^r / r!$, for $|z| < 2\pi$.

Bernoulli the related Euler polynomials and numbers[3] were proved in [Poru1975], and [Por1994b]. For exact m–covers we get from here: *System (1) is an exact m–cover with only odd moduli if and only if*

$$\begin{aligned}&\left(m-\sum_{t=1}^{k}(-1)^{a_t}b_t^r\right)E_r\\ &=\sum_{s=0}^{r-1}\binom{r}{s}E_s\cdot\left(\sum_{t=1}^{k}(-1)^{a_t}b_t^s(2a_t-b_t)^{r-s}-(-1)^{r-s}m\right)\end{aligned}\tag{7}$$

holds for every positive integer, where the Euler numbers E_r are defined through $E_r=2^rE_r(1/2)$.

This gives the following analogue of Deeba and Rodriguez recurrence for Euler numbers ([Por1994b, Theorem 4, Corollary 3]):

$$n^{2r}E_{2r}=\sum_{s=0}^{r-1}\binom{2r}{2s}n^{2s}E_{2s}\sum_{t=1}^{n-1}(-1)^{t+1}(2t-n)^{2(r-s)}\tag{8}$$

holding for positive integers r,n with $2\nmid n$.

As in the case of Bernoulli polynomials also in this case one direction of the proof of the (7) can be based on the multiplication formula for Euler polynomials. However, for Euler polynomials we have a direct analogue of multiplication formula (4) only for odd values of n. Namely,

$$E_r(x)=n^r\sum_{t=0}^{n-1}(-1)^tE_r\left(\frac{x+t}{n}\right),\quad \text{if } 2\nmid n.\tag{9}$$

For even n's the analogue of the multiplication theorem contains Bernoulli polynomials on the right hand side in the form

$$E_r(x)=\frac{2n^r}{r+1}\sum_{t=0}^{n-1}(-1)^{t+1}B_{r+1}\left(\frac{x+t}{n}\right),\quad \text{if } 2|n.\tag{10}$$

It is the aim of this paper to show that all these ideas can be extended to a more general classes of polynomials, the so called Appell polynomials.

Appell set of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ is a set of polynomials satisfying

$$f_n'(x)=nf_{n-1}(x),\qquad n=1,2,\dots\tag{11}$$

The sequence $A_n=f_n(0)$ will be called the sequence of **Appell numbers** generated by $\{f_n(x)\}_{n=0}^{\infty}$. Every Appell set of polynomials is uniquely determined by an infinite sequence of Appell numbers $A_0,A_1,A_2,\dots$ through the relations

$$f_n(x)=\sum_{r=0}^{n}\binom{n}{r}A_{n-r}x^r.\tag{12}$$

Relation (12) can be reformulated also in terms of generating series. If

$$\Phi(z)=\sum_{n=0}^{\infty}A_n\frac{z^n}{n!}$$

then

$$e^{xz}\Phi(z)=\sum_{n=0}^{\infty}f_n(x)\frac{z^n}{n!};$$

[3]Defined by the generating series $2e^{xz}/(e^z+1)=\sum_{r=0}^{\infty}E_r(x)z^r/r!$ for $|z|<\pi$.

and the relation (12) can be generalized to

$$f_n(x+u) = \sum_{r=0}^{n} \binom{n}{r} x^r f_{n-r}(u). \tag{13}$$

With

$$\Phi(z) = \frac{z}{e^z - 1}, \qquad \text{and} \qquad \Phi(z) = \frac{2}{e^z + 1}$$

we get the Bernoulli, and Euler polynomials, resp. as examples of an Appell set. The corresponding Appell numbers are the Bernoulli numbers B_n and the numbers $E_n^{(1)} = E_n(0)$ which can also be given through (see e.g. [Por1994b])

$$(r+1)E_r^{(1)} = -2(2^{r+1} - 1)B_{r+1}, \quad r = 0, 1, \dots$$

The numbers $E_n^{(1)}$ are in a close connection to the so called Genocchi numbers $G_n = 2(1 - 2^n)B_n$, more precisely, $E_r^{(1)} = G_{r+1}/(r+1)$ as easily follows. Thus any relation involving numbers $E_n^{(1)}$ can be transformed into a relation for Genocchi numbers. Moreover, due to the relation

$$E_r(x) = \sum_{t=0}^{r} \binom{r}{t} \frac{E_t}{2^t} \left(x - \frac{1}{2}\right)^{r-t}$$

every relation involving values of Euler polynomials can be rewritten into a relation involving Euler numbers, which perhaps extends the versatility of the presented ideas. Also note in this connection that these all recurrences (and the following ones) can be immediately multiplied noting that $\{(b_i - a_i - 1) \pmod{b_i}; 1 \le i \le k\}$ is also an exact n–cover with standardized offsets provided $\{a_i \pmod{b_i}; 1 \le i \le k\}$ is such a one (c.f. [Poru1998]).

Apparently, Nielsen was the first who observed that there is a hereditary aspect involved in multiplication formulas (4) and (9). He proved [Niel1923, p.54] that if a Bernoulli or Euler polynomial fulfills (4), or (9), resp., for a single value of r, then the all Bernoulli, or Euler polynomials fulfill the same identity. Unfortunately, in the general case not every Appell set of polynomials satisfies a formula analogous to (4) or (9). Carlitz extended Nielsen results in the following way (in an equivalent reformulation needed for our purposes):

Lemma 1 ([Carl1953, Theorem 1]). *Let $k > 1$ be a fixed integer; $a_{0,k}, \dots, a_{k-1,k}$ complex numbers with $a_{0,k} + \dots + a_{k-1,k} = 1$; $|l_k| \ne 0, 1$ and $b_{0,k}, \dots, b_{k-1,k}$ distinct. Then the equation*

$$\sum_{r=0}^{k-1} a_{r,k} f_n\left(\frac{x + b_{r,k}}{l_k}\right) = l_k^{-n} f_n(x) \tag{14}$$

is satisfied by a unique set of normalized polynomials $\{f_n(x)\}_{n=0}^{\infty}$ which moreover form an Appell set.

In what follows a relation of the type (14) satisfied by the all members of a set of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ will be called **multiplication formula** for this set.

2. Main results

The next Theorem gives an equivalent reformulation of (14) in terms of Appell numbers A_n.

Theorem 1. *Let $a_{0,k}, \dots, a_{k-1,k}$ be complex numbers with $a_{0,k} + \dots + a_{k-1,k} = 1$; $|l_k| \ne 0, 1$ and $b_{0,k}, \dots, b_{k-1,k}$ distinct. Then an Appell set of normalized polynomials satisfies (14) if and only if the corresponding Appell numbers with $A_0 = 1$*

satisfy the following recurrence

$$A_n = \frac{1}{1-l_k^n}\sum_{j=0}^{n-1}\binom{n}{j}A_j l_k^j \sum_{r=0}^{k-1} a_{r,k} b_{r,k}^{n-j}, \qquad n \in \mathbb{N}. \tag{15}$$

Proof. The first step of the proof, that a normalized polynomial f_n is uniquely determined by (14), can be realized by standard algebraic manipulations. Let

$$f_n(x) = \sum_{j=0}^{n} c_{n-j,n} x^j \qquad \text{with} \quad c_{0,n} = 1. \tag{16}$$

Then

$$f_n\left(\frac{x+b_{r,k}}{l_k}\right) = \sum_{j=0}^{n} c_{j,n}\sum_{s=0}^{n-j}\binom{n-j}{s}\frac{x^s b_{r,k}^{n-j-s}}{l_k^{n-j}} = \sum_{s=0}^{n} x^s \sum_{j=0}^{n-s}\binom{n-j}{s}\frac{c_{j,n} b_{r,k}^{n-j-s}}{l_k^{n-j}} \tag{17}$$

and (14) can be equivalently written in the form

$$\sum_{r=0}^{k-1} a_{r,k}\sum_{j=0}^{n-s}\binom{n-j}{s}\frac{c_{j,n} b_{r,k}^{n-j-s}}{l_k^{n-j}} = l_k^{-n} c_{n-s,n}. \tag{18}$$

We get from this identity that

$$c_{n-s,n} = \frac{1}{1-l_k^{n-s}}\sum_{j=0}^{n-s-1}\binom{n-j}{s} c_{j,n} l_k^j \sum_{r=0}^{k-1} a_{r,k} b_{r,k}^{n-j-s} \tag{19}$$

for $s = n-1, n-2, \ldots, 0$. In the case $s = n$ the hypotheses on a's only corroborates the assumption that

$$c_{0,n} = 1, \qquad \text{for every } n \in \mathbb{N}. \tag{20}$$

However, for other values of s beginning with $s = n-1$, relation (19) is actually a recurrence determining the c's with second index n uniquely and enabling us to compute them successively with the first index running from 1 to n. But $c_{n,n}$ is nothing else as A_n, and the necessity of the Theorem follows.

For the sufficiency note firstly that the differentiation obviously reproduces (14). Thus derivatives of f_n (i.e. f_r with $r \le n$) formally satisfy the same relation (14) as f_n does. For f_r with $r > n$ this can be verified using integration instead of differentiation. Here we have to show that we are able to complete correspondingly the set of coefficients of the polynomial resulting from the integration by an appropriate absolute term. To see this it is sufficient to show that (19) can be used for calculation of coefficients of f_n every n.

First of all, (11) is equivalent to

$$c_{n-(s+1),n} = \frac{n}{s+1} c_{n-1-s,n-1} \qquad \text{for } s = 0, 1, \ldots, n-1. \tag{21}$$

This identity is true for $s = n-1$. Suppose that we proved that (19) implies (21) for $s = n-1, \ldots, t+1$, i.e. suppose we proved that

$$c_{j,n} = \frac{n}{n-j} c_{j,n-1} \qquad \text{for } j = 0, \ldots, n-t-2.$$

Then (19) together with

$$\begin{aligned}\binom{n-j}{t+1} c_{j,n} &= \binom{n-j}{t+1} c_{n-(n-j),n} \\ &= \binom{n-j}{t+1}\frac{n}{n-j} c_{n-1-(n-j-1),n-1} = \frac{n}{t+1}\binom{n-j-1}{t} c_{j,n-1}\end{aligned}$$

imply that (21) is true also for $s = t$, as claimed.

Thus (19) and (20) is an equivalent reformulation of (14) (under hypotheses of Lemma 1) for a set of normalized polynomials to form an Appell set. Since the set of Appell numbers with $A_0 = c_{0,0} = 1$ uniquely determines Appell set of normalized polynomials, the result follows. □

Theorem 1 shows explicitly that the recurrence (5) of Deeba and Rodriguez is actually an equivalent reformulation of Raabe's multiplication theorem. Due to the possibility (12) to express Bernoulli polynomials in terms of Bernoulli numbers this result was actually proved by Beebee [Beeb1992] not only for exact covers of the type (2) but as quoted in (6) for general exact covers. Through relation (3) it was extended to recurrences involving general $(\mu, \mathfrak{m})$–covers in [Por1994a]. That Raabe's multiplication formula implies (5) was independently and later also proved by Howard [Howa1995]). For related numbers $E_r^{(1)}$ and allied Euler numbers E_r analogue results (as e.g. (7)) were proved in ([Por1994b]). Authors papers [Por1994a] and [Por1994b] were motivated by [Beeb1992] and results of [Poru1975]. All these results are also covered by the following generalization.

Relation (14) can also be equivalently expressed [Carl1953] in terms of the function $\Phi(z)$ as a function satisfying

$$\frac{\Phi(l_k^{-1} z)}{\Phi(z)} = \sum_{r=0}^{k-1} a_{r,k} e^{b_{r,k} z}. \tag{22}$$

With

$$\Phi(z) = \frac{z}{\alpha e^z - 1}, \quad \alpha \in \mathbb{C}$$

we get functions $\beta_n(x, \alpha)$ with complex α defined by the generating function (c.f. [Apos1952])

$$\frac{z e^{xz}}{\alpha e^z - 1} = \sum_{n=0}^{\infty} \beta_n(x, \alpha) \frac{z^n}{n!}.$$

This functions (in fact they are polynomials) represents a common generalization of Bernoulli and Euler polynomials. Namely,

$$\beta_n(x, 1) = B_n(x) \quad \text{for } n = 0, 1, 2 \dots$$

and

$$\beta_n(x, -1) = -n E_{n-1}(x)/2 \quad \text{for } n = 1, 2, 3 \dots.$$

More generally, if $\alpha \in \mathbb{C}$ is a (possibly complex) fth root of unity then with this α we can associate a series of f related functions

$$\beta_n(x, 1), \beta_n(x, \alpha), \beta_n(x, \alpha^2), \dots, \beta_n(x, \alpha^{f-1}) \tag{23}$$

for every $n = 0, 1, 2, \dots$.

If α is an fth root of unity and $m \equiv 1 \pmod f$ then the relation (22) is fulfilled for $l_m = m, a_{r,m} = \alpha^r/m, b_{r,m} = r$, with $0 \le r \le m - 1$, thereby verifying the fact that the polynomials $\beta_n(x, \alpha)$ satisfy a multiplication formula of the form (for another proof see [Poru1998])

$$\sum_{k=0}^{m-1} \alpha^k \beta_n\left(\frac{x+k}{m}, \alpha\right) = m^{-n+1} \beta_n(x, \alpha).$$

Unfortunately, polynomials $\beta_n(x, \alpha)$ do not form a normalized Appell set if $\alpha \neq 1$. This can be achieved by a small modification taking

$$\Phi(z) = \frac{\alpha - 1}{\alpha e^z - 1}, \qquad \alpha \neq 1.$$

Then we get a set of polynomials $\tau(x,\alpha)$ introduced by Euler ([Eule1755, p.487–491]; see also [Frob1910]). Obviously

$$\tau_n(x,\alpha) = \frac{\alpha-1}{n+1}\beta_{n+1}(x,\alpha), \qquad \alpha \neq 1,\ n = 0,1,2,\dots$$

and their generating function is

$$\frac{\alpha-1}{\alpha e^z - 1}e^{xz} = \sum_{n=0}^{\infty} \tau_n(x,\alpha)\frac{z^n}{n!}, \qquad \alpha \neq 1.$$

Multiplication formula discovered by Carlitz [Carl1953] for function $\tau_n(x,\alpha)$ provided α is an fth root of unity and $m \equiv 1 \pmod f$ can be now readily derived from the above lines

$$\sum_{k=0}^{m-1} \alpha^k \tau_n\left(\frac{x+k}{m},\alpha\right) = m^{-n}\tau_n(x,\alpha). \tag{24}$$

For $\alpha = -1$ this implies (9). The multiplication formula (24) through Theorem 1 gives, the following recurrence for corresponding Appell numbers $\tau_n(\alpha) = \tau_n(0,\alpha)$:

Corollary 1.1. *Let $\alpha \neq 1$ be an fth root of unity and $m \equiv 1 \pmod f$ with $m > 1$ then*

$$\tau_n(\alpha) = \frac{1}{1-m^n}\sum_{j=0}^{n-1}\binom{n}{j}\tau_j(\alpha)m^j\sum_{r=0}^{m-1}\alpha^r r^{n-j}$$

for every $n = 1,2,\dots$

Note moreover that the Appell numbers $\tau_n(\alpha) = \tau_n(0,\alpha)$ can be expressed in terms of Stirling numbers $S(n,k)$ of the second kind, where $x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\dots(x-k+1)$. Namely (see [Apos1952, formula (3.7)])

$$\tau_n(\alpha) = \sum_{j=1}^{n}\left(\frac{\alpha}{1-\alpha}\right)^j j!S(j,n).$$

As already noted Euler polynomials correspond to $\alpha = -1$, what gives the following result:

Corollary 1.2. *Let $m > 1$ be an odd number. Then*

$$E_n^{(1)} = \frac{1}{1-m^n}\sum_{j=0}^{n-1}\binom{n}{j}E_j^{(1)}m^j\sum_{r=0}^{m-1}(-1)^r r^{n-j}$$

for every $n = 1,2,\dots$.

If we substitute for Appell numbers $E_n^{(1)} = E_n(0)$ their expression in terms of Euler numbers E_n we get a special case of (7). If we express them in terms of Bernoulli numbers we get further recurrences for Bernoulli numbers (c.f. results of [Por1994b]). The above result in terms of Genocchi numbers can be found in [Howa1995, Theorem 6].

We showed in Theorem 1 that relation (14) can be equivalently reformulated into one of the form (15). The next Theorem shows that we can start even with the more general identities of the form (3).

Theorem 2. *Let*

$$\{c_t \pmod{d_t}; 1 \le t \le k\}, \tag{25}$$

be a $(\mu,\mathfrak{m})$–cover and d_0 the period of its covering function $\mathfrak{m}$. Let $\{f_n\}_{n=0}^{\infty}$ be an Appell set of polynomials defined by Appell numbers $\{A_n\}_{n=0}^{\infty}$. Let for every

$d_t, 0 \le t \le k$ numbers $a_{0,t}, \dots, a_{t-1,t}, l_{d_t}$ and $b_{0,t}, \dots, b_{t-1,t}$ are given such that for every $n = 0, 1, 2, \dots$ we have

$$l_{d_0}^n \sum_{t=0}^{d_0-1} \mathfrak{m}(t) a_{t,d_0} f_n \left(\frac{x + b_{t,d_0}}{l_{d_0}} \right) = \sum_{i=1}^{k} \mu_i a_{c_i,d_i} l_{d_i}^n f_n \left(\frac{x + b_{c_i,d_i}}{l_{d_i}} \right). \tag{26}$$

Then (26) holds for every n if and only if

$$\begin{aligned} A_n \cdot & \left[l_{d_0}^n \sum_{t=0}^{d_0-1} \mathfrak{m}(t) a_{t,d_0} - \sum_{i=1}^{k} \mu_i a_{c_i,d_i} l_{d_i}^n \right] \\ &= \sum_{r=0}^{n-1} \binom{n}{r} \cdot A_r \cdot \left[\sum_{i=1}^{k} \mu_i a_{c_i,d_i} l_{d_i}^r b_{c_i,d_i}^{n-r} - l_{d_0}^r \sum_{t=0}^{d_0-1} \mathfrak{m}(t) a_{t,d_0} b_{t,d_0}^{n-r} \right] \end{aligned} \tag{27}$$

for every n.

The proof which uses (12) and standard algebraic manipulations parallel to those of the proof of Theorem 1 is left to the reader. Note that in the proof it is not necessary to assume that the underlying Appell set is normalized. Thus it can be used for polynomials $\beta_n(x, \alpha)$. For these polynomials the identity corresponding to (3) has the form (it is interesting to note that it yields the Nielsen's relation (10) for exact covers (2) with even k):

Lemma 2. *Let $n = 0, 1, 2 \dots$ and $x \in \mathbb{R}$ and α an fth root of unit. If (1) is a $(\mu, \mathfrak{m})$–cover then*

$$\sum_{t=0}^{b_0-1} \mathfrak{m}(t) \alpha^t b_0^{n-1} \beta_n \left(\frac{x+t}{b_0}, \alpha^{b_0} \right) = \sum_{t=0}^{w} \mu_t \alpha^{a_t} b_t^{n-1} \beta_n \left(\frac{x + a_t}{b_t}, \alpha^{b_t} \right).$$

Together with Lemma 2 we get

Corollary 2.1. *Let (25) be a $(\mu, \mathfrak{m})$–cover and d_0 the period of its covering function $\mathfrak{m}$. Let α be an fth root of unity. If there exits a positive integer d such that for every $i = 0, 1, 2, \dots, k$ we have $d_i \equiv d \pmod f$ then for every n we have the recurrence*

$$\begin{aligned} \beta_n(0, \alpha^d) \cdot & \left[d_0^{n-1} \sum_{t=0}^{d_0-1} \mathfrak{m}(t) \alpha^t - \sum_{i=1}^{k} \mu_i \alpha^{c_i} d_i^{n-1} \right] \\ &= \sum_{r=0}^{n-1} \binom{n}{r} \cdot \beta_r(0, \alpha^d) \cdot \left[\sum_{i=1}^{k} \mu_i \alpha^{c_i} d_i^{r-1} c_i^{n-r} - d_0^{r-1} \sum_{t=0}^{d_0-1} \mathfrak{m}(t) \alpha^t t^{n-r} \right] \end{aligned} \tag{28}$$

For a number of specializations of this results in terms of values of Bernoulli or Euler polynomials we refer the reader to the papers quoted above.

3. Acknowledgment

The author is indebted to the anonymous referee for his comments which helped to improve the exposition of the paper.

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