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ON ARITHMETIC DENSITIES OF SETS OF GENERALIZED INTEGERS

Štefan PORUBSKÝ (*Czech Republic*)

INTRODUCTION

If A is a sequence of positive integers, then a basic relation between the lower and upper asymptotic and logarithmic densities says

$$0 \leq \underline{d}(A) \leq \underline{\ell}(A) \leq \overline{\ell}(A) \leq \overline{d}(A) \leq 1. \tag{1}$$

In the talk we shall summarize results of [3, 4, 7, 8] showing conditions

- under which a relation of type (1) remains true between generalized asymptotic and logarithmic densities,
- under which given any quadruple α, β, γ, δ of real numbers, such that $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$, there exists a set A with prescribed generalized asymptotic densities $\underline{d}(A) = \alpha$, $\overline{d}(A) = \delta$ and prescribed logarithmic densities $\underline{\ell}(A) = \beta$, $\overline{\ell}(A) = \gamma$.

To do this, we shall use an approach to arithmetic densities based on the weighted means. Moreover, the analysis will be developed on the background of the notion of arithmetical semigroups. Here under an *arithmetical semigroup* $\mathbb{G}$ we understand a free commutative (multiplicative) semigroup with identity element $1_{\mathbb{G}}$ and at most countable set $P_{\mathbb{G}}$ of generators (the so-called *primes*) of $\mathbb{G}$ endowed with a real-valued *norm* mapping $|\cdot|\colon \mathbb{G} \to \mathbb{R}$ such that (cf. [2])

- $|1_{\mathbb{G}}| = 1$, $|a| > 1$ for all $a \in \mathbb{G}$ and $a \neq 1_{\mathbb{G}}$,
- $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{G}$,
- $N_{\mathbb{G}}(x) = \sum_{\substack{|a| \leq x \\ a \in \mathbb{G}}} 1$ is finite for each real x.

There are two important classes of arithmetical semigroups. The analytical theory of the first one is developed in [2]. In this case, the analytical properties of the counting function $N_{\mathbb{G}}(x)$ are determined by the following axiom.

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Axiom A. *There exists positive constants A and δ and a constant η with $0 \leq \eta < \delta$ such that*

$$N_{\mathbb{G}}(x) = Ax^{\delta} + O(x^{\eta}).$$

The definition of the next closest class of arithmetical semigroups employs the notion of the δ-regularly varying functions. Here the function $F(x)$ defined and measurable on $\langle 0, \infty)$ is called *δ-regularly varying function*, where $\delta \in (-\infty, +\infty)$, if (cf. [9])

$$\lim_{x\to\infty} \frac{F(\lambda x)}{F(x)} = \lambda^{\delta} \quad \text{for every } \lambda > 0.$$

If we write $F(x) = x^{\delta} L(x)$, then $L(x)$ is called *slowly oscillating* (i.e. it is a δ-regularly varying function with $\delta = 0$). An arithmetical semigroup $\mathbb{G}$ will be called *δ-regular* if its counting function $N_{\mathbb{G}}(x)$ is a δ-regularly varying function (cf. [10]).

Contrary to the class of semigroups satisfying Axiom A, the theory of δ-regular semigroups does not parallel completely the expected properties known for the positive integers.

1. Density concepts and weighted means

We shall model the density concepts using the machinery of weighted means. Let $\mathfrak{m}\colon \mathbb{G} \to \mathbb{R}^{+}$ be a function defined on an arithmetical semigroup $\mathbb{G}$ and taking positive real values. For $C \subset \mathbb{G}$, let $N_C(\mathfrak{m}, x) = \sum_{|a|\leq x} \mathfrak{m}(a)\chi_C(a)$, and let

$$\sigma_x(C, \mathfrak{m}) = \frac{N_C(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} = \frac{\sum_{|a|\leq x} \mathfrak{m}(a)\chi_C(a)}{\sum_{|a|\leq x} \mathfrak{m}(a)}$$

denote the $\mathfrak{m}$-weighted arithmetic means of the indicator function χ_C of C. The numbers

$$\underline{\sigma}(C, \mathfrak{m}) = \liminf_{x\to\infty} \sigma_x(C, \mathfrak{m}) \qquad \text{and} \qquad \overline{\sigma}(C, \mathfrak{m}) = \limsup_{x\to\infty} \sigma_x(C, \mathfrak{m})$$

are called the *lower $\mathfrak{m}$-density of C* and the *upper $\mathfrak{m}$-density of C*, respectively. If $\underline{\sigma}(C, \mathfrak{m}) = \overline{\sigma}(C, \mathfrak{m})$, the common value is called the *$\mathfrak{m}$-density of C* and is denoted by $\sigma(C, \mathfrak{m})$.

The basic properties of a density concept are usually summarized by the following ones (cf. [6, p. 70]):

- **a:** density is a non-negative real number,
- **b:** finite sets have zero density,
- **c:** if $A \subset B$, then the density of A does not exceed the density of B.

To ensure that these properties are also true for the above $\mathfrak{m}$-density, we shall suppose that

A: $\mathfrak{m}$ is non-negative, i.e. $\mathfrak{m}(a) \geq 0$ for every $a \in \mathbb{G}$,
B: $\sum_{a\in\mathbb{G}} \mathfrak{m}(a)$ diverges.

Clearly, **A** implies **a** and **c**, **B** implies **b**. Property **B**, in addition to this, implies that the involved summation method is regular.

The following variant of Knopp's theorem on the convergence kernel is instrumental in the proof of relations of type (1) between two generalized densities (here $N'_C(\mathfrak{m}, x) = \sum_{|a|=x} \mathfrak{m}(a)\chi_C(a)$):

Lemma 1 ([8, Lemma 4]). *Let $\mathfrak{s}$ and $\mathfrak{m}$ be two positive functions defined on an arithmetical semigroup $\mathbb{G}$ such that*

(i) *the series $\sum_{a\in\mathbb{G}} \mathfrak{m}(a)$ diverges,*

(ii) $\displaystyle\lim_{x\to\infty} \frac{N'_{\mathbb{G}}(\mathfrak{s}, x)}{N_{\mathbb{G}}(\mathfrak{s}, x)} = \lim_{x\to\infty} \frac{\sum_{|a|=x} \mathfrak{s}(a)}{\sum_{|a|\leq x} \mathfrak{s}(a)} = 0,$ *and*

$$\lim_{x\to\infty} \frac{N'_{\mathbb{G}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} = \lim_{x\to\infty} \frac{\sum_{|a|=x} \mathfrak{m}(a)}{\sum_{|a|\leq x} \mathfrak{m}(a)} = 0,$$

(iii) *for every $a_1, a_2 \in \mathbb{G}$ such that $|a_1| \leq |a_2|$ we have*

$$\frac{\mathfrak{s}(a_2)}{\mathfrak{s}(a_1)} \geq \frac{\mathfrak{m}(a_2)}{\mathfrak{m}(a_1)}.$$

Then

$$\underline{\sigma}(C, \mathfrak{s}) \leq \underline{\sigma}(C, \mathfrak{m}) \leq \overline{\sigma}(C, \mathfrak{m}) \leq \overline{\sigma}(C, \mathfrak{s})$$

for every $C \subset \mathbb{G}$.

2. From asymptotic density to logarithmic one

The logarithmic density reflects some multiplicative properties of positive integers better than the asymptotic one (cf. [1]). In [7], we developed an idea showing how the logarithmic density can be derived from the asymptotic one. In what follows unless contrary is stated, we shall suppose that the given function $\mathfrak{m}\colon \mathbb{G} \to \mathbb{R}$ satisfies conditions **A** and **B** listed above.

The following property **M** plays a crucial role in the process of deriving a new density from that given by $\mathfrak{m}$:

M: to every $a \in \mathbb{G}$ there exists a positive real number $\widehat{\mathfrak{m}}(a)$, $\widehat{\mathfrak{m}}(a) < 1$, such that for every subset $C \subset \mathbb{G}$ having the $\mathfrak{m}$-density $\sigma(C, \mathfrak{m})$, the set $aC = \{ac : c \in \mathbb{G}\}$ also has $\mathfrak{m}$-density and $\sigma(aC, \mathfrak{m}) = \widehat{\mathfrak{m}}(a)\sigma(C, \mathfrak{m})$.

This property shows a certain dependency between the $\mathfrak{m}$-density and the multiplicative structure of $\mathbb{G}$.

The next result partially answers the question under what conditions $\widehat{\mathfrak{m}}$ also satisfies conditions **A** and **B**.

Theorem 1 ([8, Theorem 1]). *Let $\mathbb{G}$ be an arithmetical semigroup. Let $\mathfrak{m}$ satisfy conditions* **A**, **B** *and* **M**. *Let in the case, when*

$$\sum_{p \in P_{\mathbb{G}}} \widehat{\mathfrak{m}}(p) < \infty \tag{2}$$

we have

$$\sigma_x(p\,\mathbb{G}, \mathfrak{m}) = K\widehat{\mathfrak{m}}(p) \tag{3}$$

for some constants $K > 0$ uniformly in x and $p \in P_{\mathbb{G}}$. Then $\widehat{\mathfrak{m}}$ fulfills conditions **A** *and* **B**.

A slowly oscillating function $L(x)$ will be called *good* if $\lim_{x \to \infty} Z(x) = \infty$, where

$$Z(x) = \int_1^x L(y) y^{-1}\,\mathrm{d}y.$$

Note that if $\liminf_{x \to \infty} L(x) > 0$, then L is good.

The next result shows that there is a wide class of $\mathfrak{m}$-densities which together with the derived $\widehat{\mathfrak{m}}$-density fulfill the "inclusion" inequalities (1) between the asymptotic and logarithmic densities.

Theorem 2 ([8, Theorem 2]). *Let $\mathfrak{m}$ be a completely multiplicative function defined on an arithmetical semigroup $\mathbb{G}$. Let*

$$N_{\mathbb{G}}(\mathfrak{m}, x) = \sum_{|a| \le x} \mathfrak{m}(a) = x^{\delta} L(x), \tag{4}$$

with $L(x)$ a good slowly function. Then

$$0 \le \underline{\sigma}(C, \mathfrak{m}) \le \underline{\sigma}(C, \widehat{\mathfrak{m}}) \le \overline{\sigma}(C, \widehat{\mathfrak{m}}) \le \overline{\sigma}(C, \mathfrak{m}) \le 1 \tag{5}$$

for every $C \subset \mathbb{G}$.

3. Sets with prescribed upper and lower densities

The next result shows that we can prescribe generalized lower and upper densities together with the lower and upper derived logarithmic densities while inequalities (1) remain valid.

Theorem 3 ([3, Theorem 11]). *Let $\mathbb{G}$ be an arithmetical semigroup and $\mathfrak{m}\colon \mathbb{G} \to \mathbb{R}^+$ be such that*

$$\sum_{|a| \le x,\, a \in \mathbb{G}} \mathfrak{m}(a) \sim Bx^{\Delta}, \tag{6}$$

where B *and* $\Delta > 0$ *are constants. Let* $\mathfrak{m}' = \mathfrak{m}(a)/(|a|\Delta)$. *Given numbers* $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$, *there is a subset* $\mathcal{A} \subset \mathbb{G}$ *such that*

$$\underline{\sigma}(\mathcal{A}, \mathfrak{m}) = \alpha, \quad \underline{\sigma}(\mathcal{A}, \mathfrak{m}') = \beta, \quad \overline{\sigma}(\mathcal{A}, \mathfrak{m}') = \gamma, \quad \overline{\sigma}(\mathcal{A}, \mathfrak{m}) = \delta. \tag{7}$$

Condition (6) may be weakened. For example, if we define $\mathfrak{m}''(a)$ as $\mathfrak{m}'(a)/\ln(|a|)$, then analogical results can be proved along similar reasoning for this double-logarithmic density (in fact, all six upper and lower densities might be prescribed).

Following [5], it is shown in [3] that there are sets $\mathcal{A} \subset \mathbb{G}$ satisfying (7) on which an additional condition is imposed, namely that the set $R(A) = \left\{\frac{|a|}{|b|} : a \in A, b \in A\right\}$ is (topologically) dense in the set of positive real numbers $\mathbb{R}^+$.

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Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic

Email: porubsky@cs.cas.cz

URL: http://www.cs.cas.cz/porubsky