Kepler's Heritage in the Space Age

(400th Anniversary of Astronomia nova)

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In the year 1500 Europe knew less than Archimedes who died in the year 212 BC.

A. N. Whitehead

Prosthaphaeresis – a Forgotten Algorithm

Štefan Porubský

Abstract: Since the multiplication and division of large numbers is in general much more difficult to perform than addition and subtraction, various methods for reducing multiplication and division to the simpler operations of addition and subtraction were proposed to decrease the computational burden. One of the best-known such device is the use of logarithms. In this paper we summarize the history of a less known technique called prosthaphaeresis, which is based on the use of trigonometric formulas and trigonometric tables.

1 Introduction

Multiplication as a particular mathematical operation is documented in the oldest surviving mathematical texts. In its most elementary form it appears as an abbreviated form of a repeated addition. For instance, in Euclid's *Elements* (Book 7, Def. 15) we find [Euclid 2002, p.157]: 'A number is said to multiply (πολλαπλασιάσαντες) a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.' The Latin word *multiplicatio* is a compound of the words *multus* ("numerous", "many") and *plicāre* ("to fold together"). It is a Latinized form of the Greek words πολυπλασιάζειν or πολλαπλασιάζειν, used for instance by Euclid, Diophantus, Heron and Pappus.

Since the multiplication and division of big(ger) numbers is in general much more difficult to perform than addition and subtraction, various techniques for reducing the necessary computational burdens of multiplication and division were invented. The use of the logarithms is one of the best-known such techniques. One almost forgotten such device, called *prosthaphaeresis*, is based on the use of trigonometric formulae and trigonometric tables. The prosthaphaeretic method was an immediate precursor of logarithms[1] and once the idea of logarithms spread and logarithm tables

1 The Scottish astronomer and physician John Craig (d. 1620) will appear later in what follows in the role of the professor of logic and mathematics at Franfurt an der Oder. After his return home he became physician to James VI of Scotland (James I after the union of the crowns of England and Scotland). Some sources (e.g. [Gingerich 2004, p. 103]) claim that Craig was probably a member of the wedding party of James VI that was on to Denmark for the wedding of James with Princess Ann, the daughter of King Frederick of Denmark in 1590. Dreyer [Dreyer 1890, p. 203] writes that the king wished to see the spot to which the eyes of all learned men of Europe were directed, and on the 20th March he paid a visit to Tycho Brahe on Hven, arriving at eight o'clock in the morning and remaining till three p.m. Another writers describe this visit in another light, that a storm forced the party ashore on the Hven and that Craig learned at this occasion of the prosthaphaeresis method and communicated the idea to Napier (e.g. [Boyer 1968, p. 342]). The later is generally accepted to be true (cf. also [Gingerich 2004, 106]), but the former, disregarding the uncertainty of Craig's presence in the retinue, is highly improbable also due to the shortness of the visit. On the other hand, Anthony à Wood in the *Athenae Oxoniense* states that "one Dr. Craig...coming out of Denmark into his own country called upon John Neper, baron of Murcheston, near Edinburgh, and told him, among other discourses, of a new invention in Denmark (by Logomontanus, as it is said) to save the tedious multiplication and division in astronomical calculations. Neper being solicitous to know farther of him concerning this matter, he could give no other account of it than that it was by proportionable numbers." This report is, however, not generally accepted as true. On the other hand, although Napier did not mention, what kind of sources he used, in order to develop the logarithms, he may be influeced by the prosthaphaeresis, because his first logarithms were not of numbers, but were logarithms of sines. In this connection it is interesting to note that another important figure in the history of the prosthaphaeretic method Jost Bürgi (1552–1632) also invented independently the idea of logarithms.

became available prosthaphaeretic calculations faded from use in calculational aids. The name has its origin in Greek words "prosthesis" (προστεσις) and "aphaeresis", (αφερεσεις), meaning addition and subtraction. The method was proposed within the framework of the general development of trigonometry in the 16th century. The name "prosthaphaeresis" for the method used in calculations is not very felicitous since prosthaphaeresis in its original meaning of "addition-subtraction" can already be found in earlier treatises for quantities to be added or subtracted. For instance, in:

- Ptolemy's *Almagest*, as a title of the 3rd column of tables in Book III, Chapter 6 (for the Sun) and Book IV, Chapter 10 (for the Moon). (These tables are used to explain the anomaly of Sun's or Moon's motion by means of a quantity called prosthaphaeresis – later called the equation of centre – and tabulated in the columns mentioned. The quantity should be add to or subtract from the arc /angle/ at any distance of the Sun or Moon from the apogee),
- Proclus, *Catalogus codicum astrologorum Graecorum* (*Catalogue of the Codices of the Greek Astrologers*) vol. 8, part 2, Franz Cumont, Franz Boll, Wilhelm Kroll et al. (ed.), Henri Lamertin, Bruxellis 1911, p. 129,
- Viète, *Canon mathematicus seu ad triangula cum appendicibus*, Lutetiae 1579, p. 33.
- Copernicus also uses the term prosthaphaeresis several times (Chapters V, VI and VII) in his *De revolutionibus orbium coelestium*, 1543, again for a quantity which should be added or subtracted.

In what follows we shall not discuss these meanings of prosthaphaeresis.

> *Astronomy has equipped us with a soul capable of understanding Nature. Nowadays we no longer petition Nature – we command her, because we have uncovered some of her secrets and are uncovering fresh ones every day. We command her in the name of laws which she cannot repudiate because they are her own.*
>
> Henry Poincaré

1.1 The prosthaphaeretic formulas

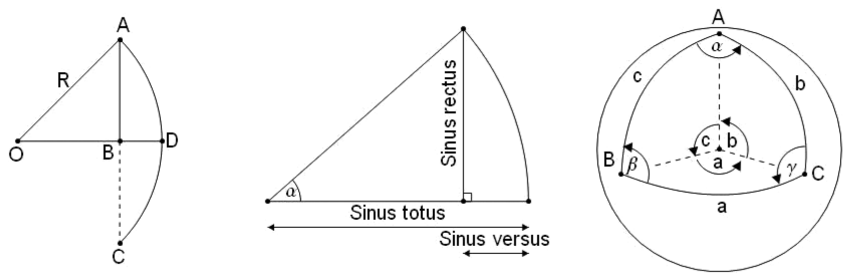
The voluminous computations of astronomers to compute positions of stars and planets at various points in space and time are based on spherical trigonometry, which relates the angles and arc lengths of spherical triangles. Typical examples are the formulas known as the *spherical law of cosines*[2] $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha$ or the *spherical law of sines* $\sin b \cdot \sin \alpha = \sin a \cdot \sin \beta$, where *a, b* and *c* are the angles subtended at the centre of the sphere by the corresponding arcs and α, β, γ are the surface angles opposite their respective arcs.[3] When all but one quantity in such a formula is known

Bürgi who lacked a formal education arrived at the invention of logarithms prior to 1587 as it was documented by Ursus and Kepler, but he published his tables (in Prague) 6 years after Napier, and so he lost the priority. One reason for this is the fact that he did not know Latin [Kepler 1606, p. 164; KGW I, 1938, p. 307] (for instance, another autodidact which also appears in our story, Ursus, in 1587 translated Copernicus' *De revolutionibus* into German for Bürgi), but Bürgi's logarithm tables are tables in a modern style. Bürgi and Ursus met at the court of William IV, Landgrave of Hesse-Kassel where Ursus was from 1586 to 1587, and Bürgi worked here as an instrument maker.

2 A variation on the law of cosines, the so-called second *spherical law of cosines* $\cos \alpha = - \cos \beta \,.\, \cos \gamma + \sin \beta \,.\, \sin \gamma \,.\, \cos a$ goes back to F. Viète.

3 The simplest way how to facilate a computation based on these formulas is to save the necessary reckoming energy once and for all in a form of properly designed tables. For instance, the fundamental relation for a right-angled spherical triangle $\sin a = \sin c \cdot \sin \alpha$ is a special case of the spherical law of sines. Regiomontanus in his *Tabula primi mobilis* compiled double-argument tables of the function arcsin, where given the angles *c* und *α* you can find the corresponding a satisfying the above equation [Braunmühl 1900, p. 122].

the unknown one can be computed by a series of multiplications, divisions, and trigonometric table references. The standard method of multiplication was long multiplication; for division it was the galley method (also called the batello or scratch method).[4] Astronomers needed thousands of such computations and these operations consumed most computational time.



It should be also noted that not only were the trigonometric tables used by astronomers of the 16th century organized in a different way, but also the terminology, notation and symbolism were different from what we use today.

There is not space enough to go into greater detail, but the basic function was the sine function. For instance, in the first book of his *De triangulis omnimodis* Regiomontanus[5] defines the *sinus* of an angle in the Hindu spirit at the beginning of Book I as follows: *When the arc and its chord are bisected we call that half-chord the right sine*[6] *of the half-arc*. As follows from this definition, the length of the segment *AB* was taken as the function of the arc *AD* and not as a function of the central angle $\angle AOB$, as we understand it today. The sine reaches its greatest value when the arc is 90° and equals the radius *R* of the circle. For this reason this value of the trigonometric sine was called *sinus totus,*[7] and the sines of other arcs were regarded as fractions of this. It was Rheticus who, for the first time, defined the trigonometric functions in terms of angles of a right-angle triangle rather than as a function of arcs and as the ratios as we know it rather than lengths as was usual in that period.

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4 Various methods for the multiplication and division of multiplace numbers were proposed in the 15th century, 16th century and in later centuries. They mostly differed from each other only by the system of notation of the intermediate computations. A. Riese (1492–1559) was one of the most influential in the process to replace the old computation in terms of counters and Roman numerals by the newer methods using pen and Hindu–Arab numerals. J. Bürgi developed independenty of Stevin the decimal notation for decimal fractions, and he also devised the multiplication scheme which we use today. He was led to these inventions in conection with computations with sines, as it is reported by Kepler (Wolf 1982, p. 15, Bürgi 1992, p. 6–7, or Frisch 1858–1871, V, p. 547).

5 Regiomontanus completed his treatise *On Triangles* in 1464, but it was published posthumously in 1533.

6 The right sine means here the *sinus rectus* (vertical sinus) what was the name of the ordinary sine function in contrast to the versed *sine* (sinus versus), which is the distance of the center of the chord to the center of the arc corresponding to the chord. The versed sine was the second most important trigonometrical function in this time and it is related to the sine by the formula vers sin $\alpha = 1 - \cos \alpha$ on a unit circle. As a function it is today practically unkown. However, this terminology was not generally used. For instance, Rheticus (1514–1576), the leading German astronomer of the first half of the 16th century, preferred to use the term *perpendiculum* for sine. In 1624 English clergyman Edmund Gunter (1581–1626) used as the first the abbreviation *sin* in a drawing and in 1634 the French mathematician Pierre Hérigone (1580–1643) used *sin* for the first time in a book.

The history around the cosine function cosine was paralel but a step delayed. Viète proposed the term *sinus residuae* for the cosine and Pitiscus called it *sinus complementi*. Gunter (1620) suggested the term *co-sinus* (*sinus complementi*).

7 The term sinus totus (total sine) was for the first time used by Gerhard of Cremona in his translation of al-Zarqali's astronomical tables.

Actually in the first four books of *De triangulis* Regiomontanus uses only one trigonometric function, namely the *sinus* [Braunmühl 1900, p. 129].[8] Copernicus a century later also works with the sine function only, which he calls the *half-chord substending double arc*.[9] Rheticus a disciple of Copernicus[10] already worked with all six trigonometric functions,[11] though under names which are no longer used.

The trigonometric identities that form the basis of prosthaphaeresis relate products of trigonometric functions to sums. Two basic trigonometric identities which were exploited in the prosthaphaeretic method convert products of sines or cosines to their sums have the following form in our modern notation:

$$\sin a \cdot \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right], \qquad (1)$$

$$\cos a \cdot \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]. \qquad (2)$$

Actually, the first formula was written in a more "homogenized" form for a + b < 90°:[12]

$$\sin a \cdot \sin b = \frac{1}{2}\left[\sin\{(90° - a) + b\} - \sin\{(90° - a) - b\}\right]$$

and so only sine tables were necessary. It was called the *first prosthaphaeretic formula*, to distinguish it from (2) the so-called *second prosthaphaeretic formula*, which was discovered later. The advantage of the second formula is that only cosine values appear there. We can easily deduced from the above formulas the next two related "mixed" identities

$$\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right], \quad \cos a \cdot \sin b = \frac{1}{2}\left[\sin(a-b) - \sin(a+b)\right].$$

which, after introducing the secant and cosecant functions, can be used for division. From these identities we can immediately obtain the reverse identities expressing addition in terms of multiplication

$$\sin a + \sin b = 2\sin\frac{a+b}{2} \cdot \cos\frac{a-b}{2}, \qquad \sin a - \sin b = 2\cos\frac{a+b}{2} \cdot \sin\frac{a-b}{2},$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2} \cdot \cos\frac{a-b}{2}, \qquad \cos a - \cos b = -2\sin\frac{a+b}{2} \cdot \sin\frac{a-b}{2}.$$

8 Although Regiomontanus defines the complementary arc (which coincides with our one for arcs less than 90º), he does not introduce a name for sine function of such argument..

9 Note that Copernicus knew of the secant which he called the hypotenusa, but no application of this function due to him is known [Braunmühl 1900, p. 144].

10 G. J. Rheticus visited Copernicus in 1539 and brought a copy of Regiomontanus' *De triangulis* to him. The copy with Copernicus' numerous marginal notes survived. Rheticus is often named as the first disciple of Copernicus, probably due to their age difference, but they studied mathematics under Rheticus guidance. Two chapters of not yet at that time published Copernicus' book *De revolutionibus* containing all the trigonometry relevant to astronomy were published in 1542 by Rheticus under the title *De lateribus et angulis triangulorum, tum planorum rectilineorum, tum sphaericorum, libellus eruditissimus et utilissimus... scriptus a clarissimo et doctissimo D. Nicolao Copernico Toronensi. Additus est Canon semissium subtensarum rectarum linearum in circulo. Vittembergae, anno 1542.*

11 It was Rheticus' tract *Canon doctrinae triangulorum* published in Leipzig in 1551, which was the first publication containing six-functions trigonometric tables. However, he yet did not use now common names for them. For instance, the function secant appears for the first time here. The tables were accurate to 7 decimal places, that is with radius 10^7, and interval from 10″ to 10″.

12 Although it is not necessary to do this today, one distingueshed the cases $a + b < 90°$ and $a + b > 90°$ at that time. For instance, the extension of (1) to the case $a + b > 90°$ is attributed to Bürgi [Cantor 1913, p. 643] and Braunmühl [Braunmühl 1899, p. 21] says that it was Bürgi who proved the case $a + b = 90°$. This later case is often attributed to Christoph Clavius (1537–1612) who published it for the first time in his *Astrolabium* (1593), pp. 179–180 [Smith 1959, p. 459]. Actually Clavius was the first who published a complete proof of (1). He also extended here the prosthaphaeretic method to secants and tangents [Smith 1959, p. 455].

These formulas turned out to be useful after the discovery of logarithms to compute sums or difference of the sine and cosine functions using logarithmic tables.[13]

1.2 The prosthaphaeretic algorithm

It should be noted that one needs to distinguish between two facts: (1) who discovered one of the prosthaphaeretic formulas and when (and perhaps /1a/, who realized that it could be used as an efficient transformation tool in formula evaluation), and (2) who was the author of the idea to apply it to make the computations easier and when was this discovery made. Before we devoted ourselves to these two questions, we show some examples of the application.

In Chapter III, 7 of *De revolutionibus* Copernicus needs the result of the product 1;10° sin 23;40°.[14] Using his sine table given in Chapter I we get sin 23;40° = 4014 (the value is 0.401414997...). Thus, 1;10° sin 23;40° ≈ 0;25,5,55° which he rounded off to 28′.

In what follows Paul Wittich (1546?–1586) will play a crucial role, but for the moment we postpone the discussion of his place in our story. On the margin of the page of his Liège copy of *De revolutionibus* containing this result he checked the result using prosthaphaeresis as follows.[15] Wittich worked with angle 22;28,16°, however, and with the value 1;12° (but sin 1;12° = 0;1,13,17° ??); moreover, he perhaps used his own sexagesimal sine tables:

90° – 22;28,16° – 67;31,44°
67;31,44° + 1;12° = 68;43,44°
67;31,44° – 1;12° = 66;19,44°
sin 68;43,44° = 55;54,47,46° (in fact, sin 68;43,44° = 55;54,44,50,26° and
sin 66;19,44° = 54;57,6,54° sin 66;19,44° = 54;57,6,52,14°)

The difference of the sines is 0;57,40,52°, half of this being, 0;28,50,26° which corresponds to the arc 0;27,32°. This is approximately the same result as Copernicus got.

It should be noted here that Copernicus uses for number notations two different systems. The angles measured in sexagesimal units are written using Roman numerals, and the lengths denoting the number of units are written using the Hindu system.[16] In his sine tables he used sinus totus $R = 10^5$.[17] From our point of view, sinus totus, whose length is a power of 10 makes the usage of the tables very comfortable if the

13 Another later important application of the last formulas appears in physics, say in the theory of vibrations.

14 The value 1;10° is the result of Copernicus' computation 50°/sin 45;17,30°, where he takes sin 45;17,30° = 7107 from his sine table. In fact, in our notation sin 45;17,30° = 0.71069716174...

15 The same computation can also be found in the *De revolutionibus* copies denoted as Wroclaw 4 or Edinburg 7 (John Craig's copy) in [Gingerich 2002].

16 In the above calculation the computations are done on the level of integers,and not fractions. In Europe decimal fractions began to be employed in the 15th century, but became slowly widely known only in the 16th century following the publication of the studies of S. Stevin. Stevin's 36-page booklet called *De Thiende* ('the art of tenths') was first published in Dutch in 1585.

17 The first decimal tables of sine, cosine, tangent and cosecant functions in Western mathematics were compiled by Giovanni Bianchini around 1450 for his astronomical calculations. Bianchini says in a letter in 1463 to Regiomontanus that the aim of this type of tables facilitaced the "task of astrologers". The advantage of the decimal system over a sexagesimal one for performing calculations was pointed out by Regiomontanus in his *Tabulae directionum* and realized in his table of tangents (*Tabula fecunda*) with $R = 10^5$ in 1467 and his third sine table with $R = 10^7$ in 1468. Before Copernicus tables from 1543 decimal trigonometrical tables were published by Apianus in 1541. Then there followed Rheticus in 1542 and 1551 with his early tables [Rosińska 1987].

computations are done in the decimal system. The astronomers were greatly aided by the Hindu-Arabic numeral system. In the 16th century the advantage of the decimal over a sexagesimal system for performing calculations became increasingly manifest. The advantages of this number system were stressed by the sophisticated arithmetic needed for scientific calculations and a wide variety of calculating tools which were developed to save the time and increase the accuracy of calculation.

A picture of the real implementation of concrete arithmetic operations in the scientific community of the 16th century can be hardly deduced from published arithmetic books, but they mirror certain general tendencies. The 16th century is actually the first century of printed books, which reflect several important transitions. Two of these are the transition from: (i) the use of the Roman symbols and methods to the use of the Hindu symbols and methods, and (ii) from theoretical arithmetic for the learned to practical arithmetic for the people. Approximately three hundred works were printed on this subject, some of which ran through many editions [Jackson 1906, p. 23].

Since the second formula (2) was probably discovered by J. Bürgi, we take an example from him [Bürgi 1992, p. 25]. Here the computation uses 10 decimal place cosine tables from *Opus Palatinum*,[18] which was available to Bürgi.

Compute $2.05518112 \times 0.749472557$:

$\cos x = 0.205518112$	$\rightarrow$	$x = 78.14016949°$
$\cos y = 0.749472557$	$\rightarrow$	$y = 41.45529021°$
$\cos (x-y) = 0.801933333$	$\leftarrow$	$x-y = 36.68487928°$
$\underline{\cos (x+y) = -0.493872963}$	$\leftarrow$	$x+y = 119.59545970°$
$\cos (x-y) + \cos (x+y) = 0.308060370$		
$\frac{1}{2}[\cos (x-y) + \cos (x+y)] = 0.154030185$	$\rightarrow$	$2.05518112 \times 0.749472557 = 1.54030185$

Clearly, if all the operations are performed with sufficiently high precision, the product can be as accurate as desired. Since there is no practical problem with the computations of the sums, differences or the average, the most sophisticated part of the whole computation is that where the tables of trigonometric functions and their inverses are used. For this reason, the accuracy of the whole computation depends to a large extent on the accuracy and detail of the trigonometric tables used.

Let us mention another application of prosthaphaeresis, again by Wittich, dated 21 October 1580. Wittich was at Uraniborg and during the Brahe's absence from his observatory, the observations of the comet of 1580 dated October 21, 22, and 26 were recorded personally by Wittich as was later certified by Jacob Monaw (1546–1603) [Rosen 1981, pp. 262–263]. The observation of October 21 ends with [Dreyer 1916, p. 130]:[19]

18 Rheticus' great treatise *Opus Palatinum de triangulis* was completed and published after his death by his pupil Valentin Otho in 1596. The treatise is of two volumes and almost 1500 pages and its second volume consists of tables of values of the six trigonometric functions (of an arc or angle) computed in steps of 10′ of arc and radius 10^{10}. In turned out that the tables in the *Opus Palatinum* contain errors in such extend that in some part they are even not usable (cf. [Braunmühl 1900, p. 221]). Actually, the whole computational work connected with compilation of these tables was based on a pre-computed table of sines with radius (sinus totus) 10^{15} in the range 0°=90° and step 45′ (for details of used method consult [Braunmühl 1900, p. 212–220]). The errors were later removed by Bartholomaeus Pitiscus (1561–1613).

19 There are three survived prosthaphaeretic calculations in connection with the observations of the comet of 1580 [Thoren 1988, p. 37] all of them employing (1).

H. 8 46 $\frac{1}{2}$ fuit cometa in azymutho 90°0′, habens altitudinem 16°34′, quasi non satis tamen certa propter nebulas

	16°34′ alt. com.		
	34 8		
	50 42	77384	
	17 34	30182	
		47202	
sinus declinationis quaesitae		23601	(13°39′ declinatio cometae apparens)

Here the computation is based on $\sin\delta = \sin\phi \cdot \sin h = \frac{1}{2}[\sin(90° - \phi + h) - \sin(90° - \phi - h)]$ with latitude $\phi = 55°52'$ and altitude $h = 16°34'$ (remember the azimuth of 90°).[20]

Additional examples of the use of the prosthaphaeretic method, which are due to Clavius and Pitiscus, can be found in [Smith 1959, pp. 455–458].

Al-Battani (850?–958) [Cantor 1894, p. 694] knew[21] the fundamental relation (2) between the three sides a, b, c and an angle α of a spherical triangle and even its other form

$$\text{vers sin}\,\alpha = \frac{\cos(b-c) - \cos a}{\sin b \cdot \sin c}$$

which is computationally more useful for it avoids the multiplication of two cosines in the numerator. However Braunmühl [Braunmühl 1900, p. 53] disclaims these facts. This result was definitely discovered by Regiomontanus when, as a young man, he studied al-Battani's *Astronomy,* and it can be found in Theorem 2 of Book V of his treatise *On triangles* in the following form (the result is stated literally rather than in symbols)

$$\frac{\text{vers sin}\,\alpha}{\text{vers sin}\,a - \text{vers sin}(b-c)} = \frac{R^2}{\sin b \cdot \sin c}\ .$$

By shortening the labors, the invention of logarithms doubled the life of the astronomers.

P. S. Laplace

2 Publish or perish

It is often claimed (for instance [Braunmühl 1900, p.105] or [Juschkewitsch 1964, p. 300]) that the Egyptian (Muslim) astronomer Ibn Yunus (ca. 950–1009) deduced a formula equivalent to the second prosthaphaeretic formula. D. A. King [King 1973, p. 360] says that this assertion is incorrect, and that this error goes back to J.-B. Delambre's [Delambre 1819, p. 112, 164] misunderstanding of the material in Chapter 15 of the Ibn Yunus' major work *Zij al-kabir al-Hakimi* (The big al-Hakim Ephemeris)[22] (for more details consult [King 1972, p. 7, 149]).

The man who not only discovered the prosthaphaeretic formulas but also who realized their labor-saving importance in connection with transformation of trigonometric formulas was the Nuremberg astronomer, mathematician, geographer

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20 The obtained result is pretty good, its exact value is 13°39′4″43‴...The usage of mentioned *Tabula primi mobilis* would probably do not save the labor, for the involved interpolation calculations also require time.

21 It is interesting to note that al-Battani work had also impact on the work the protagonists of our story Tycho Brahe and Kepler through his computations of the solar eccentricity, which by the way was better than that computed by Copernicus [Maeyama 1998].

22 The translation of the 10th Chapter of this *Zij* containing description of the method used for compilation of sine tables can be found in [Schoy 1929].

and parish priest Johann Werner (1468–1522), one of Regiomontanus' the most prominent disciples. Werner was also considered a skilled instrument maker. His *De triangulis per maximorum circulorum segmenta constructis libri V* plays a crucial role in our story. Its structure is very similar to the parts of Regiomontanus' *De triangulis* devoted to spherical geometry. Unfortunately, he did not find a publisher for it [Braunmühl 1900, p. 133] and it was not published before his death. Werner's manuscript of *De triangulis* was aquired after his death by mathematician and maker of astronomical instruments Georg Hartmann[23] (1480–1545) in Nuremberg [Björnbo 1907, p. 158]. From Hartmann it passed into Rheticus' hands in 1542 ([Cantor 1913, p. 454], [Danielson 2006, p. 96]) during his visit of Nuremberg. Who saw the manuscripts besides them after Werner's death is not very clear.[24] Some sources claim that Rheticus intended to use it as the basis for his own large canon on triangles over which he worked a long time. Rheticus' entire library (including Copernicus' autograph of *De revolutionibus*) went after his death in 1576 to his disciple Valentin Otho (also written as Otto). On the death of Otho (he died in 1603 in Prague) the entire Rheticus library went to Jacob Christmann (1554–1613) [Cantor 1913, p. 603] an eminent German astronomer (and orientalist about whom Cantor [Cantor 1913, p. 597], says that since 1609 he occupied the first chair of the Arabic language in Europe. This is not true, because the first such position was held by the orientalist and astronomer Guillaume Postel (1519–1581) at the Collège de France a half century before. For Postel's possible role in the transfer and relationship between the work of Copernicus and the work of the Arabic-writing astronomers preceded him, see Saliba 2007.

Let us note that Werner's Propositio II of *De motu octavae sphaerae* (1522) contains a reference to the application of prosthaphaeresis in simplifying the determination of a stellar coordinate, although without giving procedural details [Björnbo 1907, p. 155]. Björnbo [Björnbo 1907, p. 155] deduces from some passages of the collection of Werner's treatises published in 1514 that Werner knew clearly how to apply the prosthaphaeretic formula to the spherical law of cosines ($\cos\alpha = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$) at latest in 1514, and and not earlier than 1505.

Eneström [Eneström 1902] reports that Rheticus published two of Werner's treatises (one of them the *De triangulis*) in 1557 in Cracow. However the publication was not found in any library in Europe. Only a booklet containing a 10-page introduction written by Rheticus was found in the University library in Cracow, but with none of Werner's text. Rheticus' text was probably meant as an introduction to Werner's books [Björnbo 1907, p. 159].

In 1611 Christmann published the book *Theoria Lunae* in Heidelberg. He mentiones Werner's treatise, and that Werner demonstrated the method there using three

23 He was vicar of Sebalduskirche in Nuremberg, and the discoverer of the inclination of the magnetic needle. For a report obout some surviving Hartmann's astrolabes see [Lamprey 1997].

24 Bjönbo [Björnbo 1907, p. 158] writes that Peter Apian (1495–1552) published in the appendage to his book *Radius astronomicus cum quadrante novo meteoroscopii longe utilissismus* on meteroscopy a compilation of the first two Werner's book on triangles. However, North [North 1966, p. 62] says, that no such work is listed by F. van Ortroy, Apian's bibliographer, and continues: In fact Apian abstracted an appreciate part of Werner's first three books. … It was the third book, on the manifold applications of Werner's first instrument, which marked out Werner's work as unusual and which Apian used without acknowledgment in his *Astronomicum Caesareum*. … Werner's book, like Apian's, are largely concerned with the problem of simplifying the tedious calculations of spherical astronomy. As such, Werner's was the more systematic approach and, from the mathematical point off view, the more valuable. Werner's method was to show how – in the manner already explained – any right-angled triangle could be seolved, given sufficient information. He then applied the method to the solution of standard astronomical problems, where necessary resolving triangles which are not right-angled into right-angled component triangles.

examples [BRAUNMÜHL 1900, p. 136] of how a product of two sines can be replaced by a sum, as Werner really did. This was actually the first "public notice" of the existence of the yet unpublished Werner's treatise. Christmann showed here one of Werner's applications of (1) in a method of how to find the astronomical longitude of Spica Virginis from the obliquity of ecliptic, declination and latitude (Copernicus in principle solves the same problem in II,3 of *De revolutionibus,* however without prosthaphaeresis and less elegantly than Werner).

Johann Praetorius says in *Cod. lat. monac. 42101* (folio 18), written around 1599, that the formula (somewhat simplified by a student)

$$\cos a = \tfrac{1}{2}[\sin(90° - b + c) - \sin(90° - b - c)]\operatorname{vers}\sin(180° - \alpha) + \sin(90° - b - c)$$

which can be used for determination of side a from sides b, c and the included angle α was invented by Werner [BRAUNMÜHL 1900, p. 137]. In 1569, Praetorius, one of Otho's teachers, spent x some time in Cracow in close contact with Rheticus [DANIELSON 2006, p. 189]. Christmann and Praetorius' comments can be summarized by saying that Werner was aware of the importance of formula (1) for the transformation of trigonometric identities in connection with their numerical evaluations, but it does not give an answer to whether he also arrived at the idea of applying the formula to the multiplication of general numbers [BRAUNMÜHL 1900, p. 137].

We see that despite the fact that Werner's manuscript was not published, its content was known to some persons. To see who might have profited from it, it is important to reconstruct its itinerary and the sojourns of the people tied to the history of prosthaphaeresis. Rheticus is known to have come to Wittemberg in 1537. In October of 1538 he left on tour to visit key scholars in southern Germany (supported by a letter written by Melanchthon). He met Peter Apian, Johannes Schöner, Philip Imser and others and returned to Wittemberg in May of 1539 [DANIELSON 2006, p. 31, 39]. After his decision to visit Copernicus, we find the protestant Rheticus in late May of 1539 at the 'remotest corner of the earth' in Frauenburg [DANIELSON 2006, p. 41]. He returned to Wittemberg in October of 1541. In 1542 he was appointed as professor of higher mathematics at University of Leipzig. In 1545 he took another leave of absence and undertook a journey to Italy to visit Cardano, but his visit to Italy in 1545 and 1546 was an unmitigated failure [DANIELSON 2006, p. 122]. The initial intention to be away from Leipzig only for one year ended so that Rheticus was absent from Leipzig all in all for more than 3 years. In 1547 during his travels his health broke down. In 1548 he was back in Leipzig, where was elected to the position of dean on October 13 [DANIELSON 2006, p. 132]. In April 1551 he was forced to flee, due to the accusation of having a homosexual affair with one of the students. In May 5, 1551 the Leipzig city council transferred the case to the university with a request to confiscate his personal property. He flew through Chemnitz to Prague where he continued his study of medicine (which he began to study in Zurich in 1547 after his recovery). On April 11, 1552 he was banished from the University of Leipzig for 101 years [DANIELSON 2006, p. 149]. He had now permanently left his life and possessions behind in Leipzig [DANIELSON 2006, p. 156]![25]

In the winter 1553–54 he continued his study of medicine in Breslau (Wroclaw) in Silesia. In 1554 he settled down in Cracow and practised medicine for about 20 years. For reasons that are not fully understood, Rheticus left Cracow in 1572 and travelled to the region known at Scepusia, southwest of the High Tatra Mountains,

25 For information about inventories of Rheticus' possesions left in Leipzig consult [DANIELSON 2006, Chap. 12].

to the Upper Hungarian cathedral town Cassovia.[26] He took many (though not all) of his relevant documents along with him when he left Cracow [Danielson 2006, p. 188]. Rheticus died in December 1574 in Cassovia.

What happened with the Werner manuscript after it was in possession of Christmann is unclear. A handwritten copy of it, but without figures, was found by A. A. Björnbo in the Vatican library as a part[27] of a gift of Queen Christina the only surviving legitimate child of Swedish king Gustaf II Adolf. The manuscript was deposited as Codex Regensis latinus 1259, d. h. Nr. 1259 of the Regina Svecise collection. Björnbo published it in [Björnbo 1907].

It is not seemly for a professor of mathematics to be childishly pleased about any shortening of the calculations.

Maestlin to Kepler

3 *Artificium Tychonicum*[28] or *negotium Wittichianum* or … ?

3.1 Corpora delicti

There are several important documents which helped at least partly (but not definitely) to reconstruct the chains of events and to reveal the role of some persons connected with the main historical significance of the prosthaphaeretic formulas, namely their role as a calculation tool. The first one is the Werner's manuscript which we have already discussed. The second one was a thorough inspection of all survived copies of Copernicus' manuscript of *De revolutionibus.* For more details about this interesting endavour we refer the reader to [Gingerich – Westman1988], [Gingerich2002] and [Gingerich 2004]. Last but not least it was the discovery of Brahe's manual of trigonometry *Triangulorum planorum et sphaericorum praxis arithmetica*.[29]

3.2 A mysterious wandering mathematician

The German mathematician and astronomer Paul Wittich[30] is one of the crucial personalities in our story despite the fact that he had never published anything. *Well born, and probably well off, Wittich was an enigmatic character whose roots remain mostly obscure. … We now know that Wittich became a kind of itinerant humanistic tutor to men who valued and practiced astronomy in a variety of contexts* [Gingerich – Westman

26 Impression that Cassovia (Košice in Slovak, Kaschau in German, or Kassa in Hungarian) was a part of Scepusia is not correct. It lies 20 km form the south-east corner of Scepusia. Scepusia (Spiš in Slovak, Spisz in Polish, Zips in German and Széplak in Hungarian) is a region in north-eastern Slovakia with common boarder with Cracow province on the Polish side of the common Slovak-Polish boarder. Scepusia is a part of Slovakia now, but in that time both lied in upper Hungary. As a region it has a very interesting history and it is one of the German settlement regions strongly influenced by German newcomers (the so-called German colonization) coming to this region in several waves from the 13th century on. The first documents collection dating from 12th to 17th century from this region can be found in the four volumes treatise *Analecta Scepusii sacri at profane I.–IV*. (Vienae 1773, Posonii et Cassovia 1778) by Carolus Wagner (1732–1790).

27 The manuscript is listed as Codex Reginensis latinus 1259, d. h. Nr. 1259 of the Regina Sueciae collection. The text has the title Nr. 1259 *Joannis Verneri Norimbergensis De triangulis sphaericis libri IV*. Cod. ex. Papyro 4to, anno 1495. However the text does not contain any time data at all, and therefore no support for the year 1495.

28 Frisch 1858–1871, II, p. 439, note 94.

29 This manual was intended to be incorporated in the proposed treatise *Teatrum astronomicum*, but early Tycho's death prevent him to write it. The manual was never printed and was found in the only existing copy in Prague and published in a photographic copy edition in 1886 by Studnička in Prague [Studnička 1886]. Mathematically it is based on Regiomontanus' *De triangulis* and Copernicus' treatise on trigonometry.

30 O. Gingerich extrapolated from some facts that Wittich was born in 1546 in Breslau. He died in Vienna in 1586 while working for the Emperor Rudolph II.

1988]. Unfortunately, the most standard information about him are often unknown and could only be estimated against the background of the dates of the known events. We know that he matriculated at the University of Leipzig in the summer of 1563. We know that he then matriculated at Wittenberg in June 1566 and in Frankfurt an der Oder in 1576. However, no record was found that a degree was awarded to him by either of these universities. Two comments should be made here.

Tycho, who also matriculated at Leipzig in 1562 (at 15-years old), also visited Wittenberg in 1566, 1570 and 1575. He does not specify on which ocaasion he met Wittich. Rosen thinks it was 1575 but gives no evidence [Gingerich – Westman 1988, p.16]. His first visit here was at the same time that Wittich was here in 1570, but apparently they did not remember each other.

Secondly, as already mentioned, John Craig was professor in Frankfurt an der Oder in 1576. That Wittich and Craig knew each other well follows from Craig's letter to Tycho dated May 1589 [Gingerich 2002, p. 290]. It was here where Craig copied the mentioned example of the prosthaphaeretic method into his copies of *De revolutionibus*. When Wittich died Craig praised him saying, 'If you require mathematical demonstrations, when others do not suffice for the occasion, then I turn to those of Wittich' [Gingerich – Westman 1988, p. 11–12].

Wittich arrived on Hven around the end of July 1580 with a letter of recommendation from Hagecius. He helped Peter Flemhose to observe the mentioned comet which appeared in October 1580 in Tycho's absence. One of the calculations connected with its observation we presented above (Wittich was nearsighted and was therefore more interested in the theoretical aspect of astronomy). The visit to Uraniborg must have been an exciting one not only for Wittich: Tycho held nothing back as he explained the novel star-sights and scales on his quadrants, sextants, and armillary spheres. They toured the library with its thousands of books and its giant celestial globe, and they swapped notes on their ingenious trigonometrical methods for Tycho also from the viewpoint [Gingerich 2004, p. 111].

When Wittich left Hven he took with him a copy of Apian's *Astronomicum Caesareum* as a gift from Tycho with the following inscription in Latin on the title page, dated on Saturday 29 October 1580: *To Paul Wittich of Wratislava, friend and fellow lover of mathematics.*

By 1584 we find Wittich in Kassel. Here Wittich worked with Bürgi and obviously passed on what he knew in Hven. Regardless of the outflow of know-how from Hven, this was very disappointing for Tycho also from the viewpoint that Wittich promised to return, what he never did.

It is not known where Wittich found the first prosthaphaeretic formula. Was the intermediary between Paul Wittich and this knowledge a certain Johannes Wittich from Breslau, a relative of Paul, who corresponded with Rheticus [Gingerich – Westman 1988, p. 10–11], was it his uncle Balthasar Sartorius, an aquaintance of Rheticus [Gingerich – Westman 1988, p. 31], or was he in a direct contact to Rheticus?

3.3 Much ado about prosthaphaeresis

The method was first published in 1588, along with formula (1) and formula (2) in a small treatise, *Fundamentum astronomicum*, by Nicolaus Ursus[31]. The first publication not citing the source is today usually accepted as a priority claim. At that time this question was not handled so carefully by authors as it is today That priority is

31 Ursus considered only the cases $a + b < 90°$ and $a + b > 90°$ [Smith 1959, p. 459].

not the case here follows for us from already mentioned facts. At that time, however, Ursus' book evoked a stormy reaction from Tycho, who claimed the idea of prosthaphaeresis entirely as his own.[32] Tycho was extremely antagonistic towards anybody who claimed priority (not only) for prosthaphaeresis. The question of why Brahe should have made any claim to a role in the discovery of the prosthaphaeretic method has not been satisfactorily answered, because it is generally believed that it was Wittich who brought the method to Hven. Longomontanus, who was from 1589 Brahe's chief assistant, attributes [Longomontanus 1622, p. 10] prosthaphaeresis to Wittich and Tycho with the argument that it cannot be found in the treatises of the Arabs or of Regiomontanus [Wolf 1982a, p. 56].

Ursus' reply to Tycho's reactions and accusations was both personal (for details cf. Gingerich 2004, p. 115) and factually based, saying that he had learned the method during the aforementioned stay from 1586 to 1587 at the court of William IV, and that it was Wittich who bought the formula (1) there albeit without a proof. The proof was promptly supplied by Bürgi, who became so motivated that he in turn also discovered formula (2).

Note that Ursus[33] was at Hven four years earlier in 1584. Already in that time, Tycho suspected him of snooping around in his library, and Tycho even organized conspiracy actions against Ursus to prove him a spy [Gingerich 2004, p. 113]. Ursus went to Cassel in early 1586, so that Ursus and Wittich did not meet at Cassel (it at all), for Wittich was dead at that time [Thoren 1988, p. 35].

The question of whether it was Tycho's idea to apply prosthaphaeretic formulas for computational purposes is hard to answer. The truth is that we can find the prosthaphaeretic method in his *Triangulorum... praxis*; e.g. in *Dogma VI sphaericorum*, where two sides and the enclosed angle are given, the third side being given by

$$\begin{aligned}\cos a &= \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha \\ &= \tfrac{1}{2}[\cos(b-c) + \cos(b+c)] + \tfrac{1}{2}[\cos(b-c) - \cos(b+c)] \cos \alpha\end{aligned}$$

Note that two multiplications are avoided here, while the third one is left in. Similarly, in *Dogma IX sphaericorum* when the three sides are given, an angle is found by

$$\cos \alpha = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

where again the numerator and denominator could be simplified, but the division had to be done [Dreyer1916, p.128]. However the only copy of *Triangulorum... praxis* is dated 1591 and there is no evidence that pre-copies which were freely available in tables in 1580, when Wittich was there, also contained these formulas.

There is one additional interesting fact to note. Wittich owned four copies of Copernicus' *De revolutionibus*, which was certainly unusual because books were expensive at that time. Strong evidence exists in Tycho's correspondence that Wittich had carried the *De revolutionibus* containing his mathematical methods along to Hven on his 1580

32 From historical point of view recall the well known Kepler's faux pas at the beginning of his career. "Young beginner" Kepler after reading the above mentioned Ursus' book, but seeking at the same time Tycho's patronage sent a laudatory letter to Ursus, at that time imperial mathematician to Rudolph II, praising him as one of the best mathematicians and astronomers in Europe. Ursus did not reply the letter, but republished it in the preface to his work *De astronomicis hypothesibus*, 1597 (cf. [Rosen 1946] for more details).

33 Ursus started his career as an assistant to Eric Lange, a Danish nobleman and relative of Tycho Brahe.

visit [Gingerich 2004, p. 107]. After Wittich's death his possessions had gone to his sister living in Breslau. Surprisingly, Tycho made several serious attempts to buy Wittich's copies of *De revolutionibus.* The reason for this unusual activity is a mystery. Was it the fact that they contained both details of Wittich's planetary system[34] and traces of his prosthaphaeretic computations, and that their possession would bring him advantages in all these priority claims? Hagecius tried twice in 1589 and 1595 to purchase them but without success. It was Longomontanus who bought then them for Tycho in 1598. Tycho got three copies. One of them survived as a part of Tycho's estate and historians wrongly attributed Wittich's inscriptions to Tycho (one copy is in Liège, one in Wroclaw and one in the Vatican library).[35] By inspection of all survived copies Gingerich proved that the handwriting is Wittich's and not Tycho's (for more details consult Gingerich – Westman 1988, or Gingerich 2002).

The story about the priority fight in the matter of the prosthaphaeresis method is very complex and has many "black holes" and mysterious issues, as for instance, a possible appearance of the cosine-product formula (2) in Tycho's manual before Ursus' publication. We recommend the reader to read the papers given in the references to make his/her own picture about it, and to find a motivation to fill some of them, eventually.

3.4 Post scriptum

There is one additional argument against Tycho's mathematical creativity, namely the third multiplication in his *Dogma VI*. There is a simple way of avoiding it, which cannot escape the attention of a skilled mathematician. On the other hand it also shows Bürgi's mathematical expertise and dominance over other users of the prosthaphaeretic method. The idea is very clever. After the application of (1) to the first two factors in the triple product we again get products of trigonometric functions. Bürgi proposed to put $\sin b \cdot \sin c = \frac{1}{2}[\cos(b - c) + \cos(b + c) = \cos A$ for a suitable value of A. Then (1) gives $\cos a = \frac{1}{2}[\cos(b - c) + \cos(b + c)] + \cos(A - \alpha) + \cos(A + \alpha)$, a formula without any sine or cosine products [Cantor 1913, p. 643], [Bürgi 1992, p. 8–9].

We have not mentioned here certain other important persons, such as Melchior Jöstel and David Fabricius. The list of scholars who had written about prosthaphaeresis is longer that those mentioned here; e.g. G. L. Frobenius (1634), the Jewish astronomer Emanuel Porto of Padua in his *Porto astronomico* (1636) or Joseph Del Medigo (born in Crete in 1591 and died in Prague in 1655), who gave an exposition of the method in his *Mayyan Ganim* (Garden Fountain) published in Amsterdam in 1629 [Lewittes 1932], etc.

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34 Wittich's and Ursus' Capellan geoheliocentric models were, by the way, another subtopic in the controversies Tycho vs. Ursus and Tycho vs. Wittich.

35 In connection with the destiny of Werner's manuscript it is interesting to read the following lines [Gingerich – Westman 1988, p. 22–23]: "The Vatican first edition may well have passed through Tycho's hands to Rudolph II. The Swedish general Königsmarck captured the royal castle in Prague in July 1648, and sent the Rudolphine library to Queen Christina. When Christina abdicated and went to Rome, she took with her the *De revolutionibus* now in the Vatican library, and it seems likely that the source of her copy was the loot from Prague." For more details about the Swedish war loot during the Thirty Years' War consisting not only from 30 containers with books of Rudolphine library and more than 300 mathematical instruments seized solely in Prague castle and the role of Queen Christina in the course of these events consult [Callmer 1977].

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