New necessary and sufficient conditions on (a_i, m_i) in order that $x \equiv a_i (\text{mod } m_i)$ be a covering system

Štefan Porubský and J. Schönheim

Abstract

A covering system is a set of congruences $x \equiv a_i (\text{mod } m_i)$, $i = 1, \dots k$, such that every integer satisfies at least one of them.

A new necessary and sufficient condition in order that a given set of congruences $x \equiv a_i (\text{mod } m_i)$ be a covering system is established.

We show that (4) are such conditions.

For exact covering systems they are reduced to (5).

The connection of these conditions to known ones such as those [3] based on Bernoulli polynomials and those [8] based on cosets of $Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_k}$ are studied.

1. Introduction

A *covering system* CS is a set of congruences

$$x \equiv a_i (\text{mod } m_i) \qquad i = 1, 2, \dots, k \tag{1}$$

such that every integer satisfies *at least* one of them. The a_i's are called offsets and are supposed to be standardized to $0 \le a_i < m_i$.

The concept of CS as defined above has been introduced by P. Erdös in the thirties. For definitions and more details on early results see [2,6,7,10]. The particular case when "*at least*" is replaced by "*exactly*" the CS is called to be exact (ECS).

The simplest necessary condition for (1) to be a CS is

$$\sum_{i=1}^{k} \frac{1}{m_i} \ge 1 \,. \tag{2}$$

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This well known result can be obtained easily by counting arguments. It has been observed, on the one hand, that there are CS's with $\sum_{i=1}^{k} \frac{1}{m_i} = 1 + \delta$ for arbitrarily small positive δ, on the other hand that an arbitrarily large sum is not sufficient.

These facts suggest that it is impossible to look for some necessary and sufficient conditions involving exclusively the magnitudes of the moduli.

Condition (2) reduces to $\sum_{i=1}^{k} \frac{1}{m_i} = 1$ for ECS's. In that case even necessary and sufficient conditions have been obtained [2], they consist of infinitely many equalities involving besides the moduli also the offsets. Namely, for $s = 0, 1, 2, 3, \ldots$

$$\sum_{i=1}^{k} m_i^{s-1} B_s\Big(\frac{a_i}{m_i}\Big) = B_s(0) \tag{3}$$

where $B_s(x)$ is the s-th Bernoulli polynomial.

As elegant conditions (3) look like they are less efficient than the definition itself in order to check whether any system (1) is an ECS or not. All the more so in the CS case one cannot expect to find a more efficient necessary and sufficient condition than that provided by the definition. That motivates the way our results are formulated in the next section. They involve both the moduli and the offsets.

2. Results

We need some definitions and notations. Following [8] consider the group $G = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_k}$ and its subgroup H generated by the element $(1, 1, \ldots, 1)$. Suppose the elements of H are written in the order obtained putting successively

$$n = 0, 1, \ldots, N - 1 \text{ in } n(1, 1, \ldots, 1)$$

where $N = [m_1, m_2, \ldots, m_k]$.

Definition.

A coset C of H in G is called *good* if every element in C has at least one zero component. If every element in C has exactly one zero component the C is called *very good*.

Theorem 1. *The following three statements are equivalent*

1. *The set of congruences (1) is a CS.*
2. *The element $(a_1, a_2, \ldots, a_k)$ of G is a member of a good coset of H.*
3. *For $x = 0, 1, 2, \ldots, N - 1$*

(4) $$\sum_{i=1}^{k}\frac{1}{m_i} > \sum_{i=1}^{k}\frac{a_i+x+1}{m_i} - \sum_{i=1}^{k}\frac{a_i+x}{m_i}$$

where $a_i + x + 1$ and $a_i + x$ are standardized $(\text{mod } m_i)$, *while the sums are real sums.*

Theorem 2. *The following three statements are equivalent*

1. *The set of congruences (1) is an ECS*
2. *The element $(a_1, a_2, \dots, a_k)$ of G is an element of a very good coset of H.*
3. $\sum_{i=1}^{k}\frac{1}{m_i}$ *and for $x = 0, 1, \dots, N-1$*

(5) $$\sum_{i=1}^{k}\frac{a_i+x+1}{m_i} - \sum_{i=1}^{k}\frac{a_i+x}{m_i} = 0\ .$$

Observation 1. Equalities (5) are expressing the fact that the left side of (3) with $s = 1$ is the same for all members of the coset containing $(a_1, a_2, \dots, a_k)$. Moreover this is true also for the general polynomial of degree one.

3. Proofs

3.1 Proof of Theorem 1.

First we prove 1.$\Longleftrightarrow$2.

Suppose statement 2 holds and $v = (a_1, a_2, \dots, a_k)$ is an element of a good coset C. Consider the elements $h_0, h_1, \dots, h_{N-1}$ of H ordered as mentioned and choose $v = v_0$ as a coset leader of $C = \{v_0, v_1, \dots, v_{N-1}\}$. Furthermore, denote the components of v_n by $v_n = (b_{n_1}, b_{n_2}, \dots, b_{n_k})$. Then

(6) $$v_n = h_n + v_0 \qquad (n = 0, 1, \dots, N-1)$$

or componentwise $b_{n_i} = n + a_i$. By assumption, for some i, say $i = t$, $b_{n_t} = 0$, thus $0 \equiv n + a_t$ i.e. $n \equiv -a_t (\text{mod } m_t)$, meaning that every $n \in \{0, 1, \dots, N-1)$ satisfy one of the congruences

(7) $$x \equiv -a_i (\text{mod } m_i)$$

so (7) is a CS. But then by a simple argument (1) also is a CS.

Suppose statement 1 holds and (1) is a CS. Then (7) is also a CS. The coset C having a leader $v_0 = [a_1, a_2, \dots, a_k)$ is good. Indeed again consider (6) for given $n \in \{0, 1, \dots, N-1\}$.

If n satisfies the congruence $x \equiv -a_t (\mathrm{mod}\ m_t)$ for some $t \in \{1, 2, \ldots, k\}$ then (6) written componentwise gives $b_{n_t} = n + a_t \equiv 0 (\mathrm{mod}\ m_t)$.

Finally prove 2 $\iff$ 3. By the standardization above, for every integer $x \in \{a_1, \ldots, N-1\}$ and $i \in= \{1, 2, \ldots, k\}$

$$\frac{a_i + x + 1}{m_i} - \frac{a_i + x}{m_i} = \begin{cases} \frac{1}{m_i} & \text{for } x \neq m_i - 1 - a_i \\ \frac{1}{m_i} - 1 & \text{for } x = m_i - 1 - a_i \end{cases}$$

hence if and only if $x \neq m_1 - 1 - a_i$

$$\sum_{i=1}^{k} \frac{a_i + x + 1}{m_i} - \sum_{i=1}^{k} \frac{a_i + x}{m_i} = \sum_{i=1}^{k} \frac{1}{m_i} .$$

Otherwise the sum is smaller.

Observe that $x = m_i - 1 - a_i$ means that in member v_{x+1} of the coset C with leader $(a_1, a_2, \ldots, a_k)$ the i-th component is zero. Therefore, (4) implies statement 2 and vice versa.

3.2 Proof of Theorem 2.

We omit the proof of Theorem 2 since it is in the lines of the proof of Theorem 1.

4. Final Remarks

1. Theorem 2 in view of Observation 1 provides an affirmative answer to the question whether the sufficiency of condition (3) can be reduced to a finite number of equalities.

2. The results of this paper can be related also to the theory of covering functions; see [4]. We shall elaborate this in another paper.

3. For other classes of systems of congruences than CS and ECS, such as the *M-time systems* and various *disjoint systems* theorems like Theorem 1 and 2 can be established, completing results due to J. Beebee [1] and Z.W. Sun [9].

Štefan Porubský and J. Schönheim

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Department of Mathematics, Institute of Chemical Technology, Techniká 5, 166 28
Prague 6, Czech Republic
E-mail address: porubsks@vscht.cz

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty
of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel
E-mail address: shanan@math.tau.ac.il